CS130 : Computer Graphics
Curves (cont.)

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Blending Functions

Blending functions are more convenient basis than monomial basis

- "canonical form" (monomial basis)

$$\mathbf{f}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3$$

- "geometric form" (blending functions)

$$\mathbf{f}(u) = b_0(u)\mathbf{p}_0 + b_1(u)\mathbf{p}_1 + b_2(u)\mathbf{p}_2 + b_3(u)\mathbf{p}_3$$

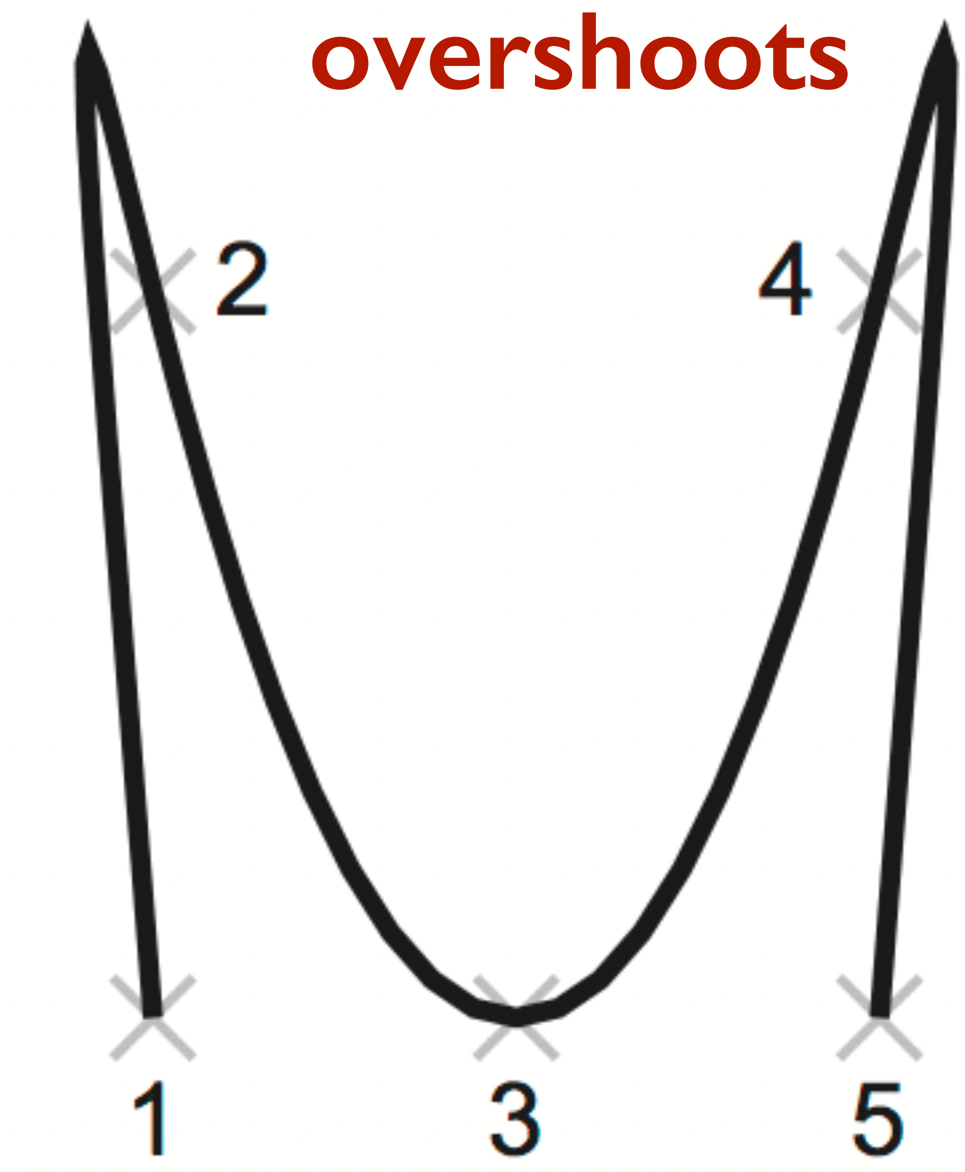
Interpolating Polynomials

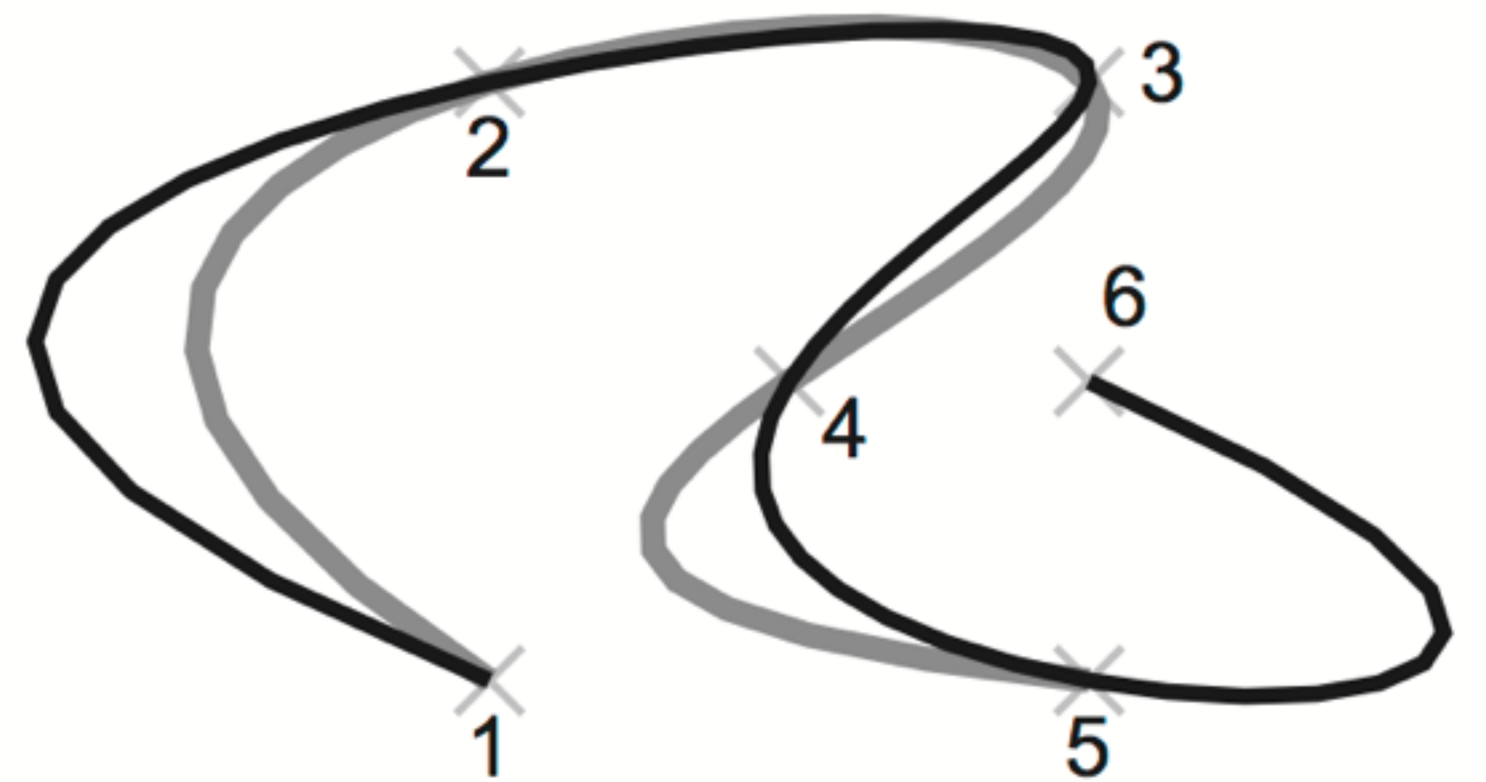
Interpolating polynomials

- Given n+1 data points, can find a unique interpolating polynomial of degree n
- Different methods:
  - Vandermonde matrix
  - Lagrange interpolation
  - Newton interpolation

higher order interpolating polynomials are rarely used

overshoots





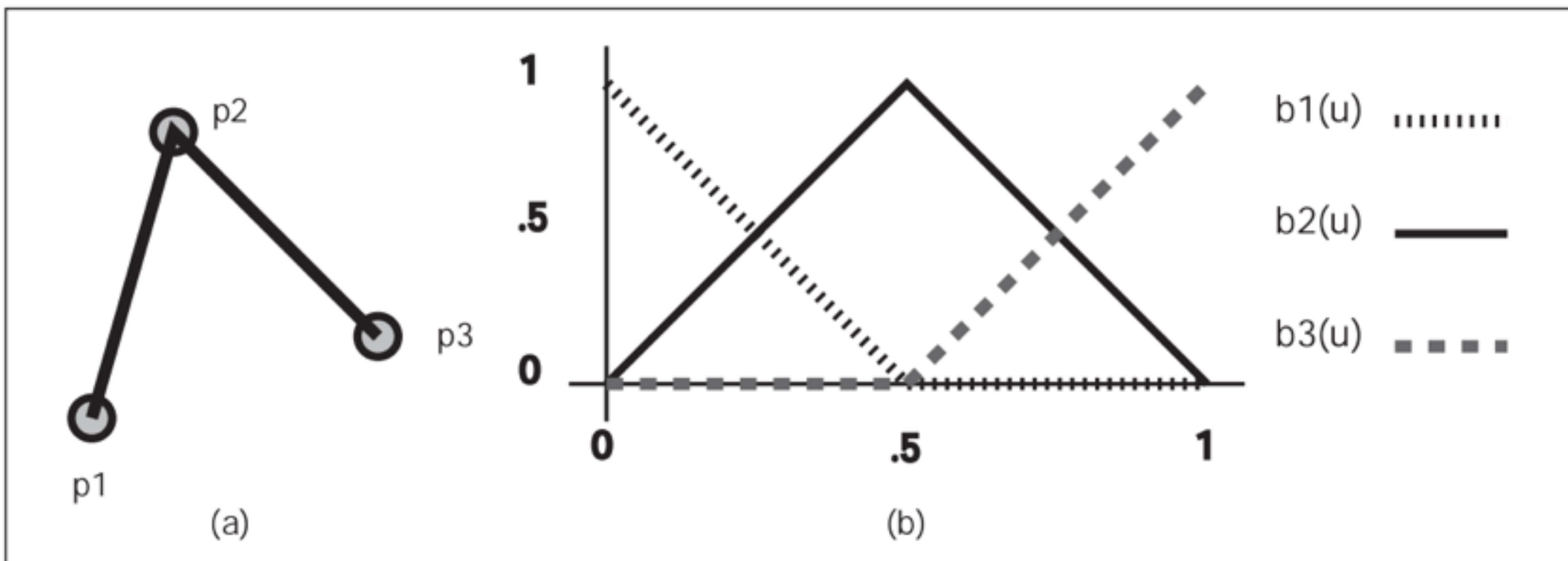
non-local effects

4th order (gray) to 5th order (black)

Piecewise Polynomial Curves

Example: blending functions for two line segments

$$\mathbf{f}(u) = \begin{cases} \mathbf{f}_1(2u) & u \leq 0.5 \\ \mathbf{f}_2(2u-1) & u > 0.5 \end{cases}$$



Cubics

$$\mathbf{f}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3$$

- Allow up to C^2 continuity at knots
- need 4 control points
  - may be 4 points on the curve, combination of points and derivatives, ...
- good smoothness and computational properties

We can get any 3 of 4 properties

1. piecewise cubic
2. curve interpolates control points
3. curve has local control
4. curves has C2 continuity at knots

Cubics

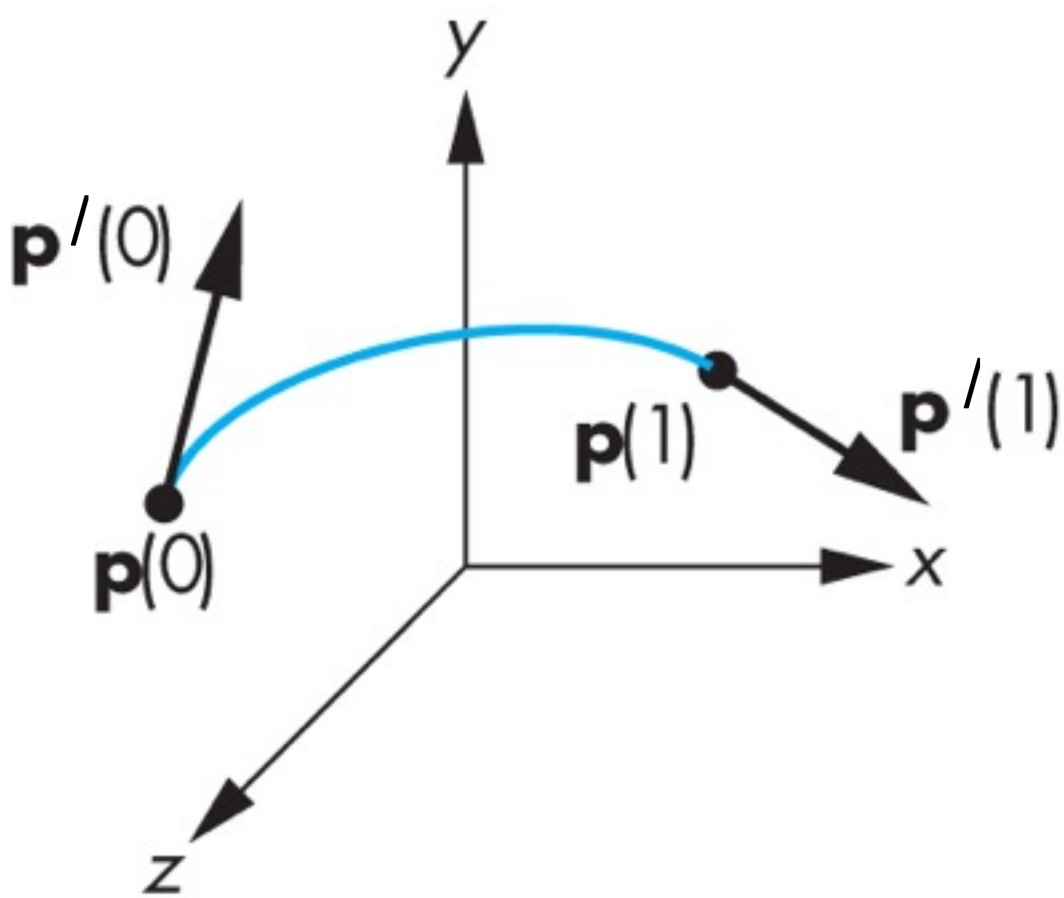
- Natural cubics
  - C2 continuity
  - n points -> n-1 cubic segments
- control is non-local :(
- ill-conditioned x(

Cubic Hermite Curves

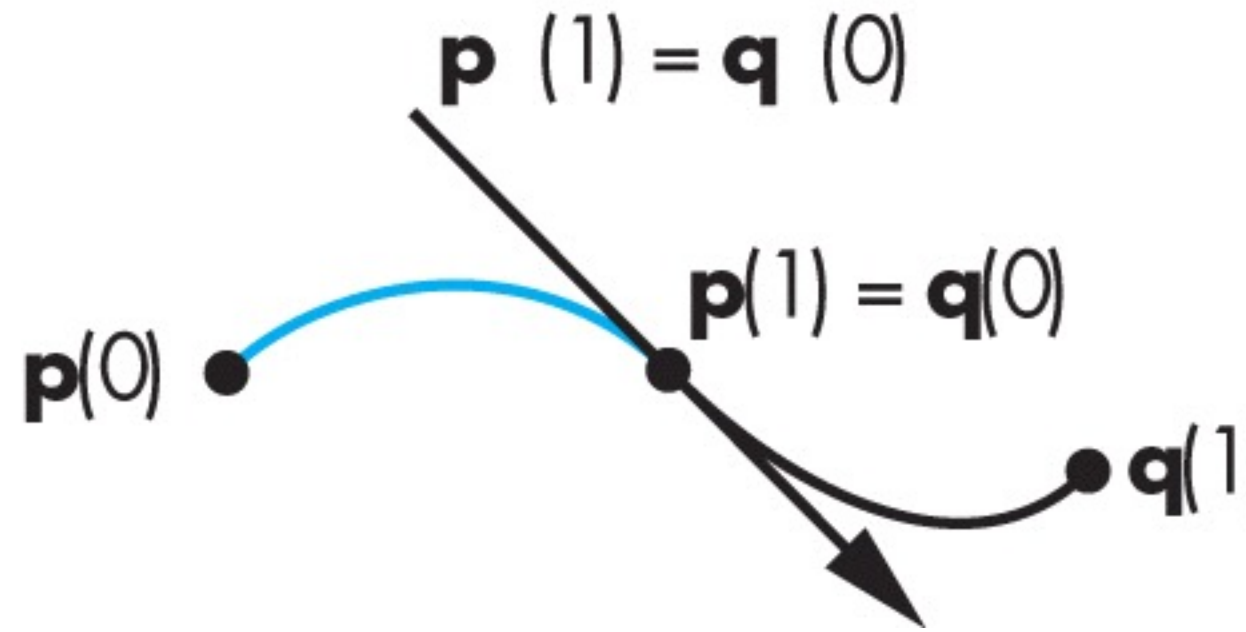
- C1 continuity
- specify both positions and derivatives

Cubic Hermite Curves

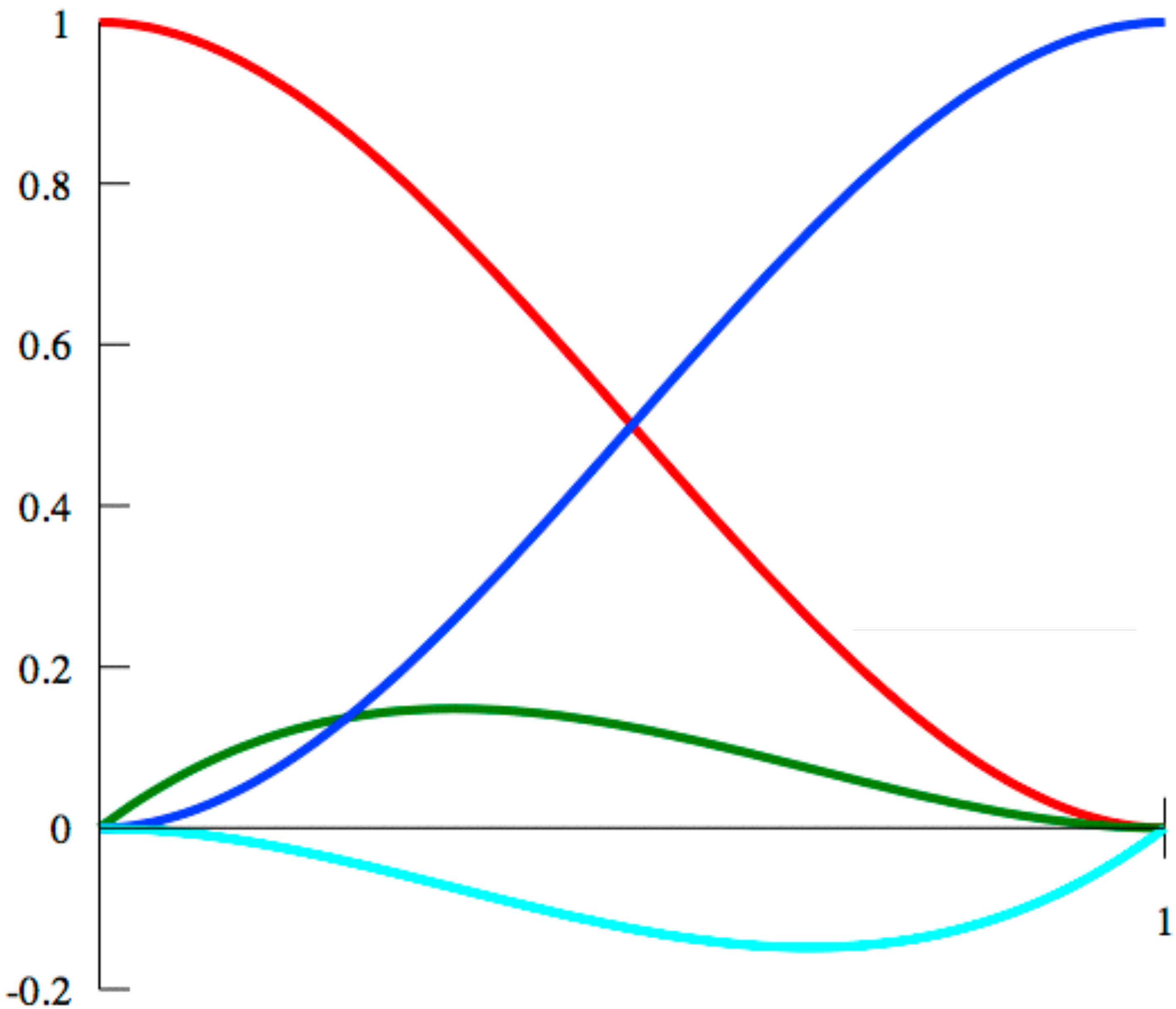
Specify endpoints and derivatives



construct curve with C^1 continuity



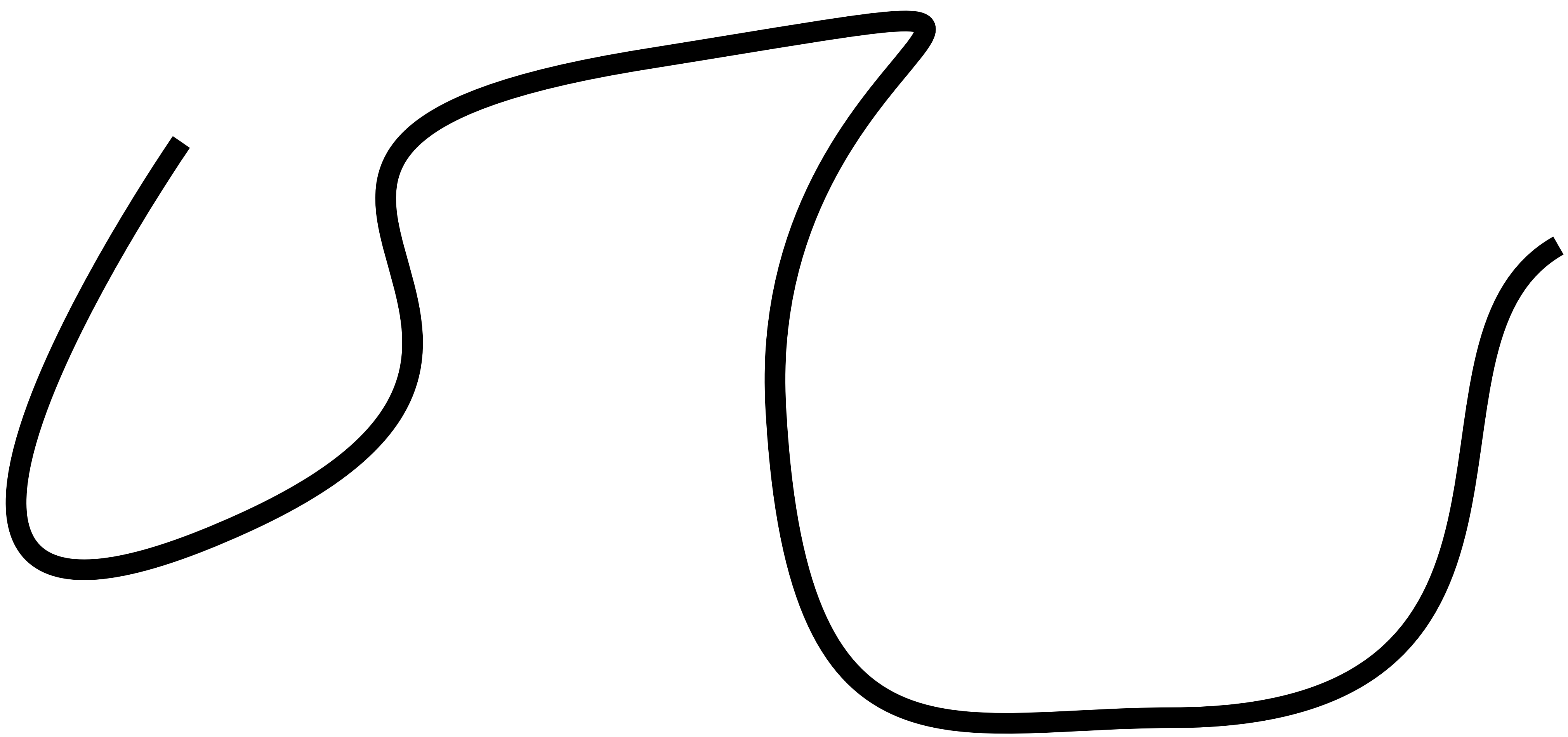
Hermite blending functions



[Wikimedia Commons]

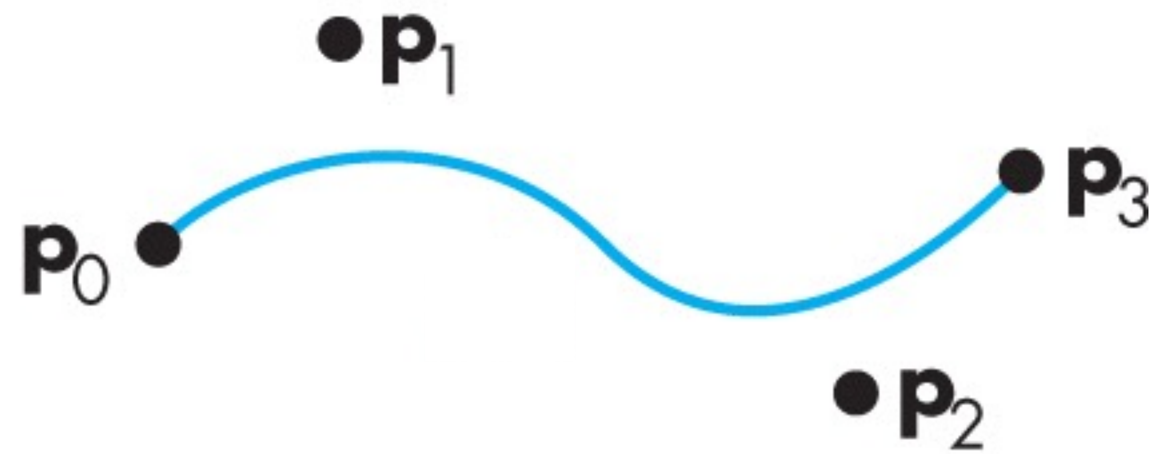
$$b_0(u) = 2u^3 - 3u^2 + 1$$
$$b_1(u) = -2u^3 + 3u^2$$
$$b_2(u) = u^3 - 2u^2 + u$$
$$b_3(u) = u^3 - u^2$$

Example: keynote curve tool



Interpolating vs. Approximating Curves

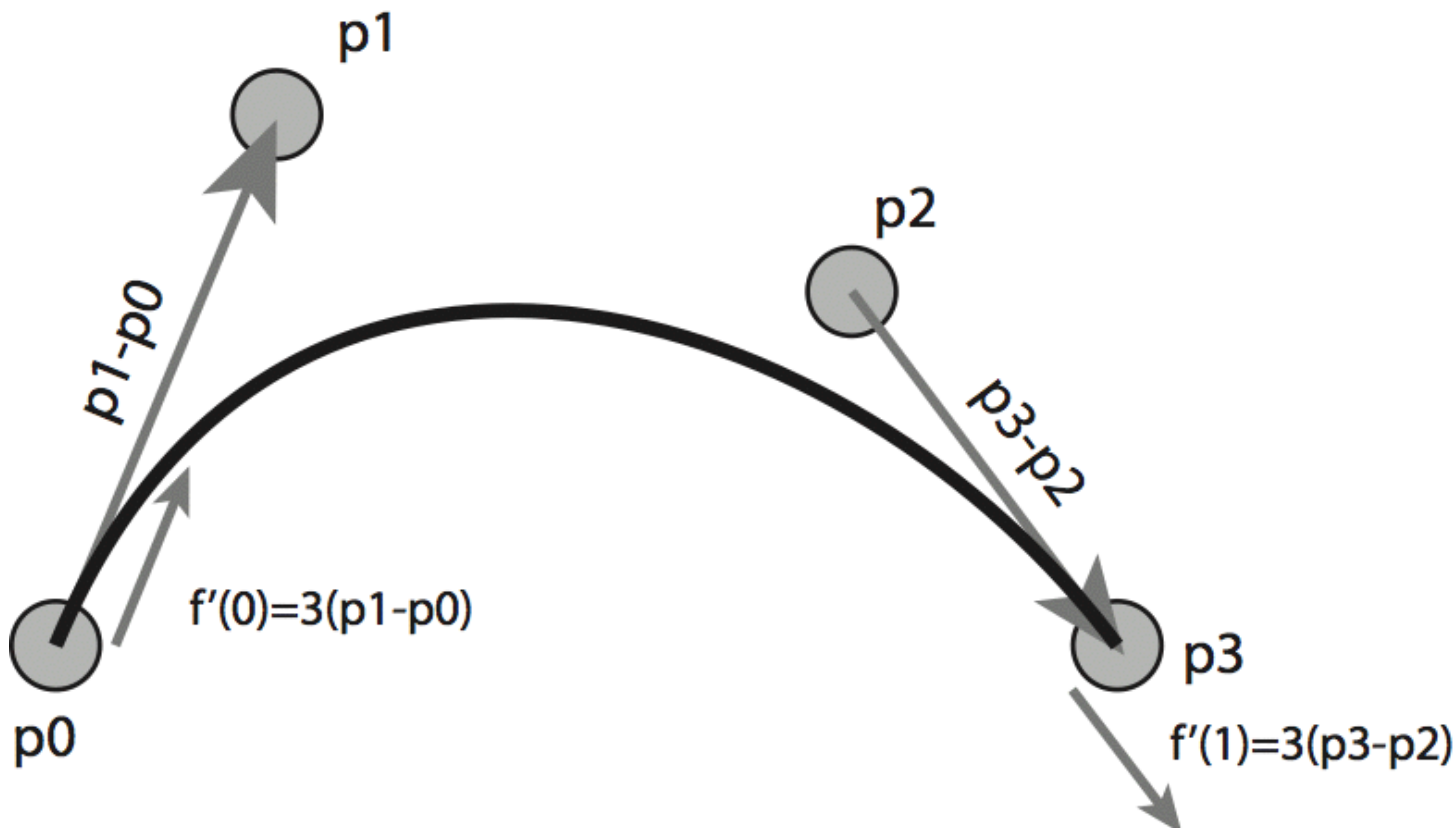
Interpolating



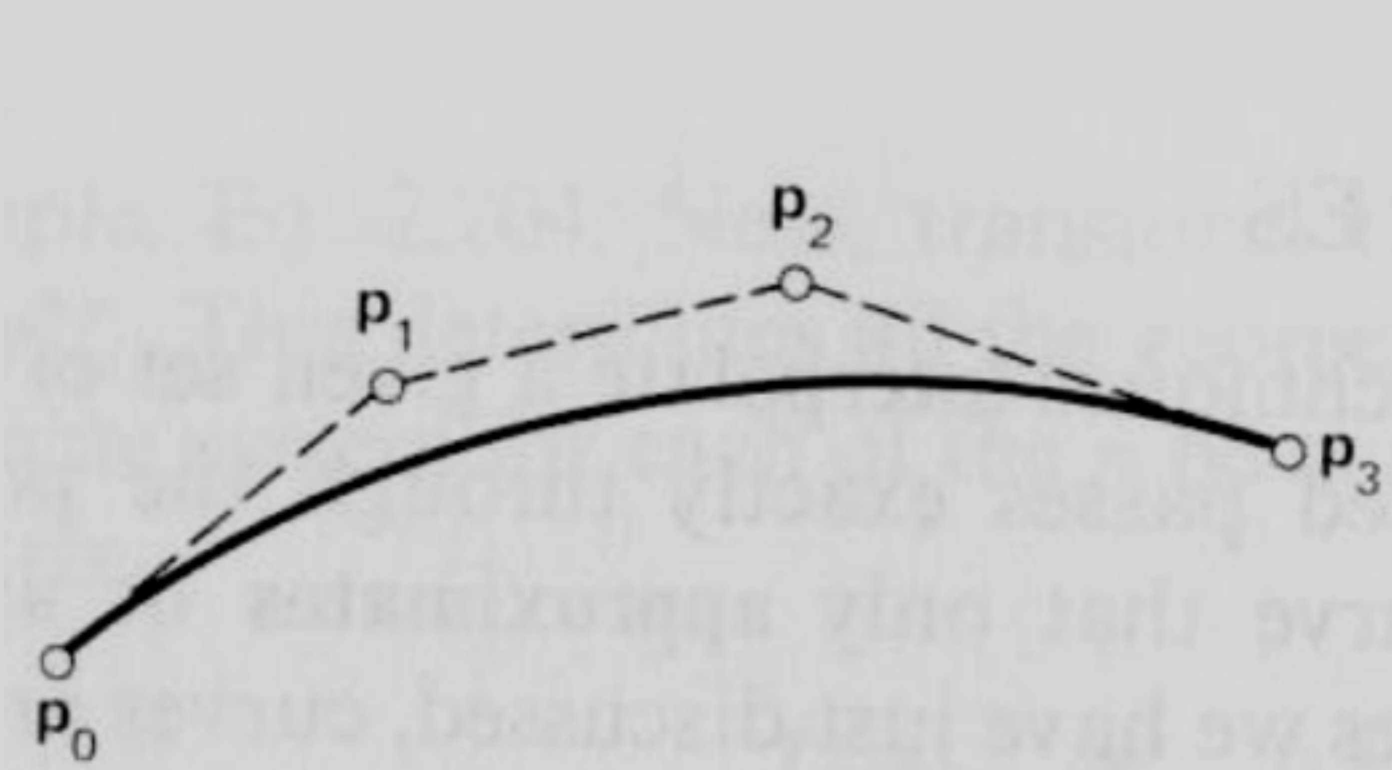
Approximating
(non-interpolating)

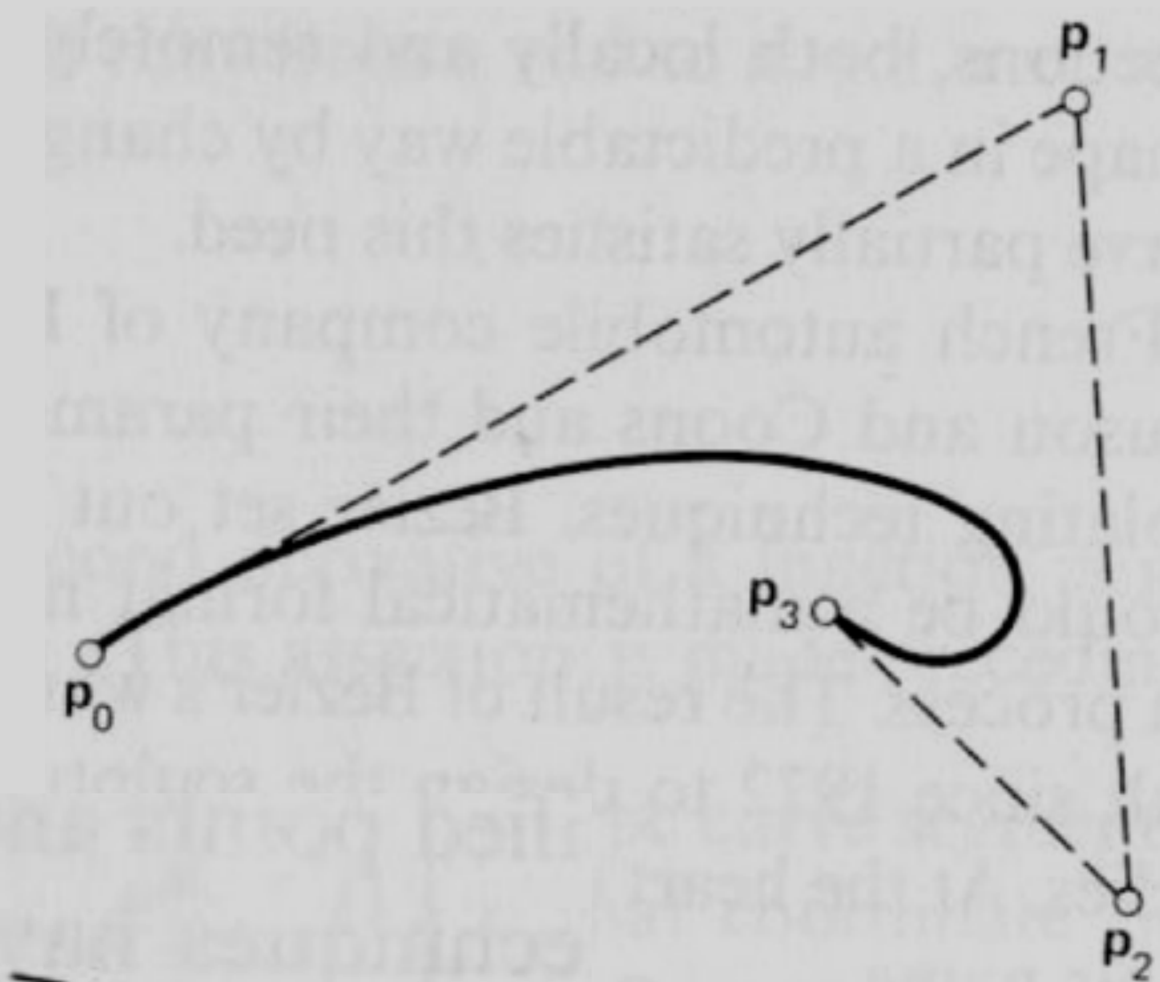
Cubic Bezier Curves

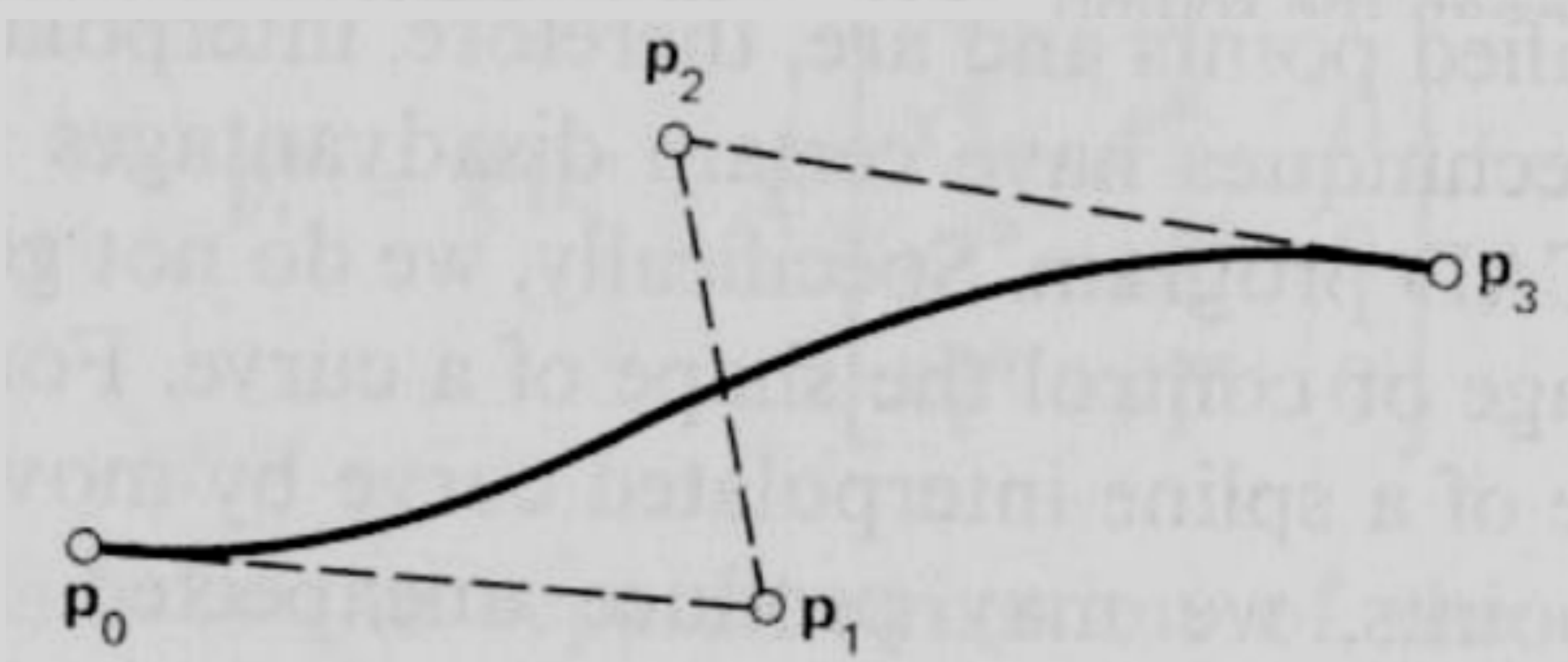
Cubic Bezier Curves



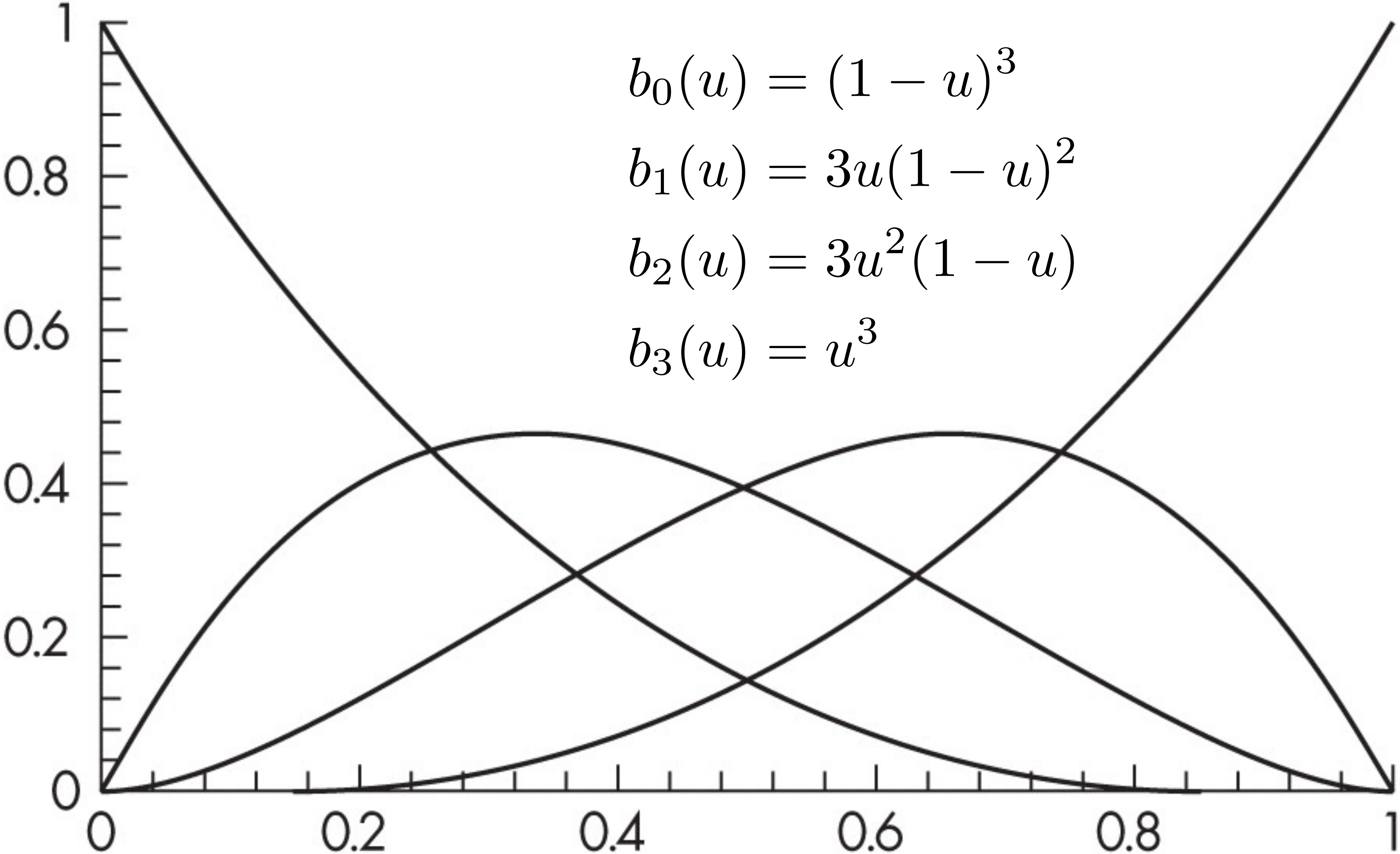
Cubic Bezier Curve Examples







Cubic Bezier blending functions



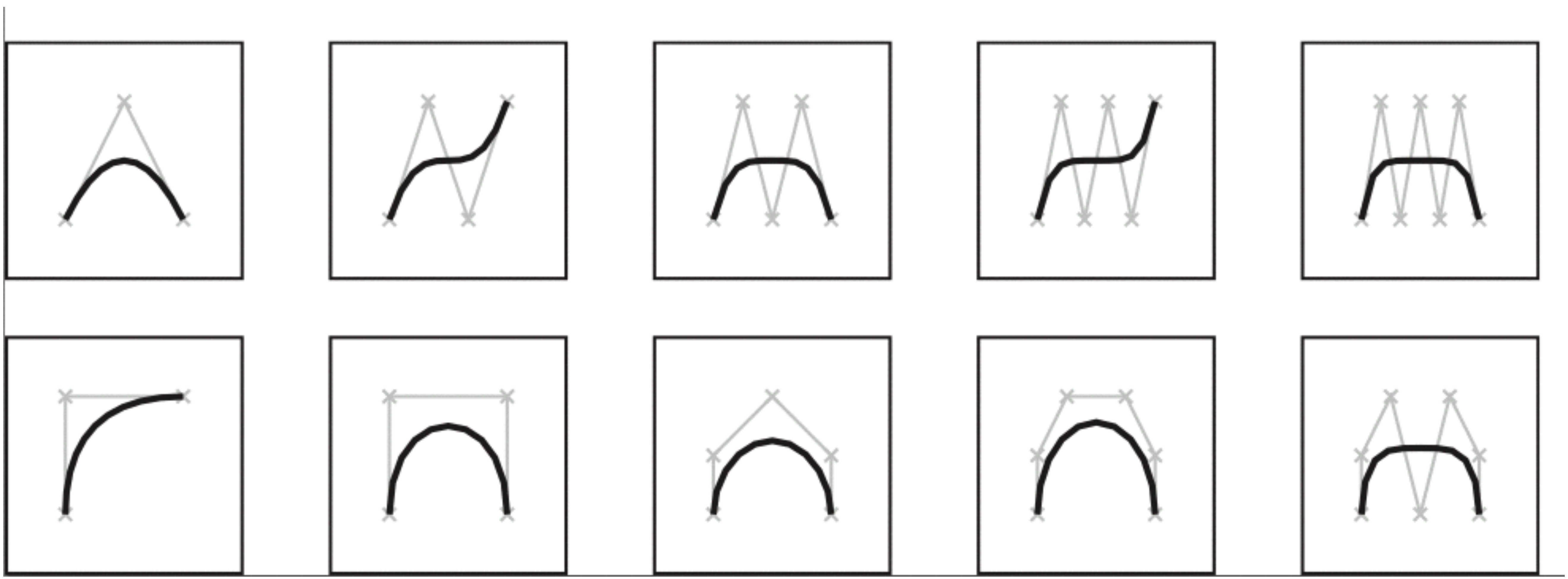
$$b_0(u) = (1-u)^3$$

$$b_1(u) = 3u(1-u)^2$$

$$b_2(u) = 3u^2(1-u)$$

$$b_3(u) = u^3$$

Bezier Curves Degrees 2-6



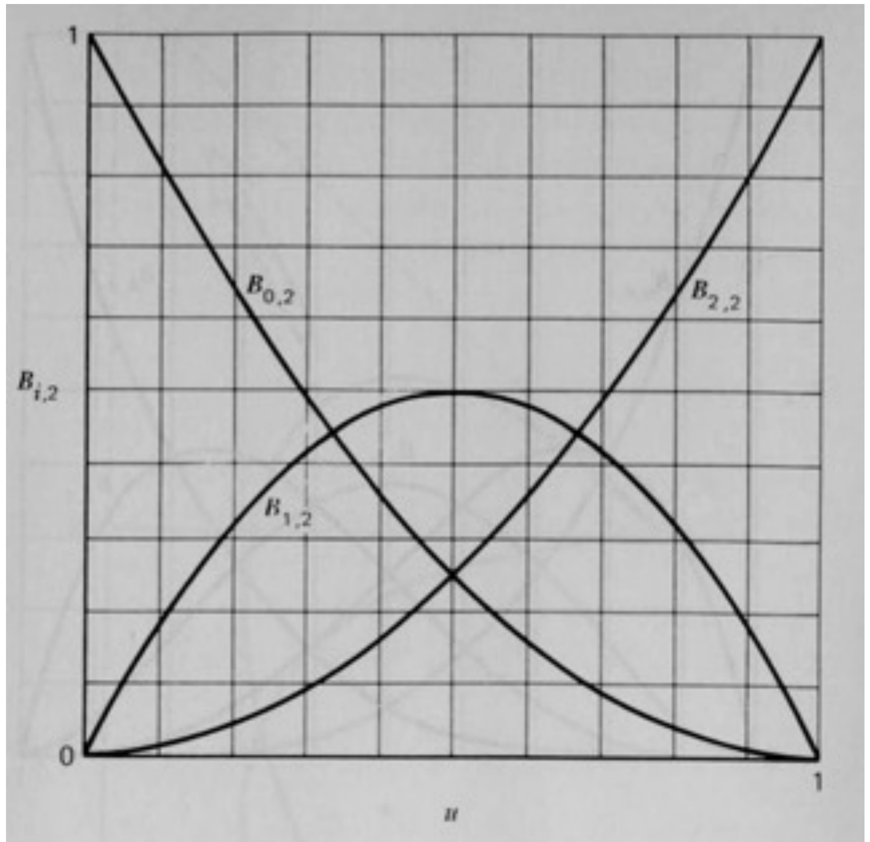
Bernstein Polynomials

- The blending functions are a special case of the Bernstein polynomials

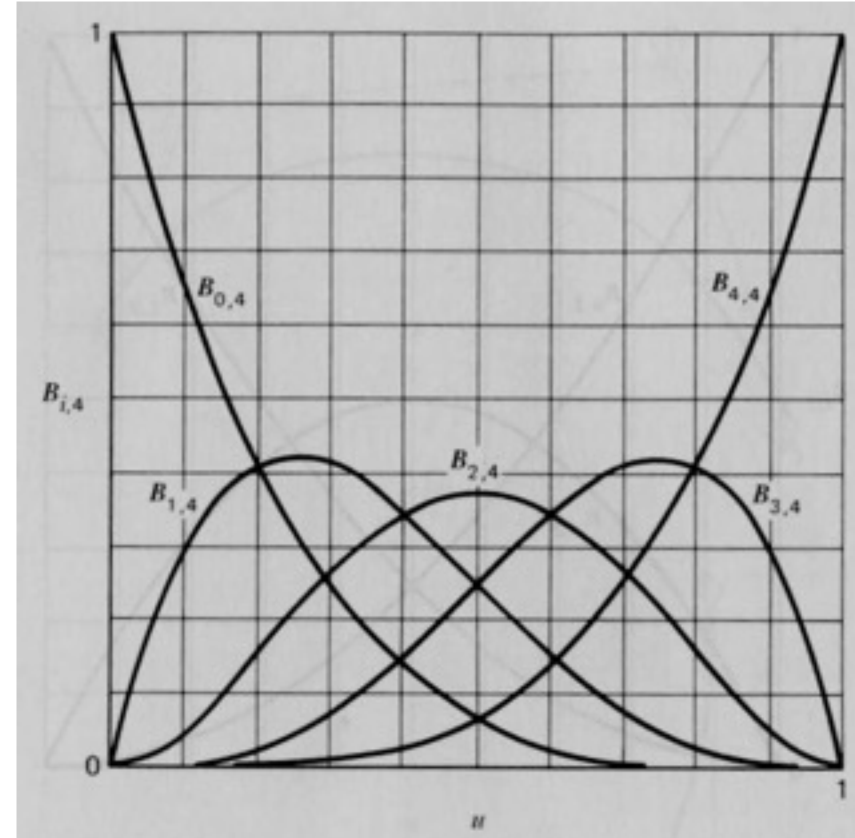
$$b_{\mathrm{kd}}(u) = \frac{d!}{k!(d-k)!} u^k (1-u)^{d-k}$$

- These polynomials give the blending polynomials for any degree Bezier form
  - All roots at 0 and 1
  - For any degree they all sum to 1
  - They are all between 0 and 1 inside (0,1)

n = 3



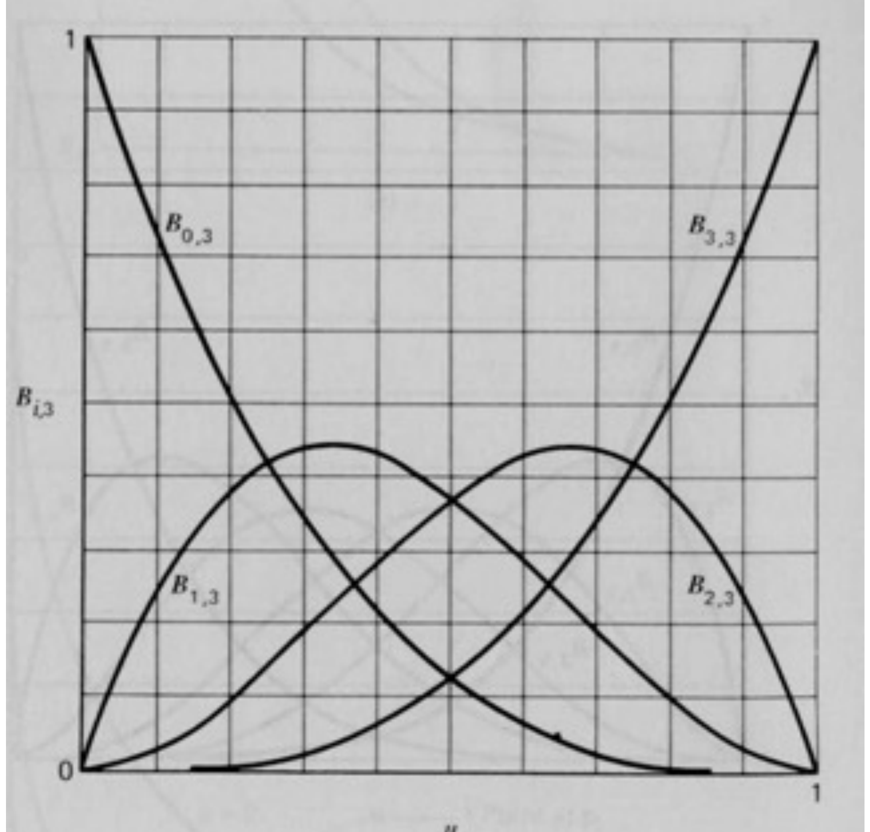
(a)



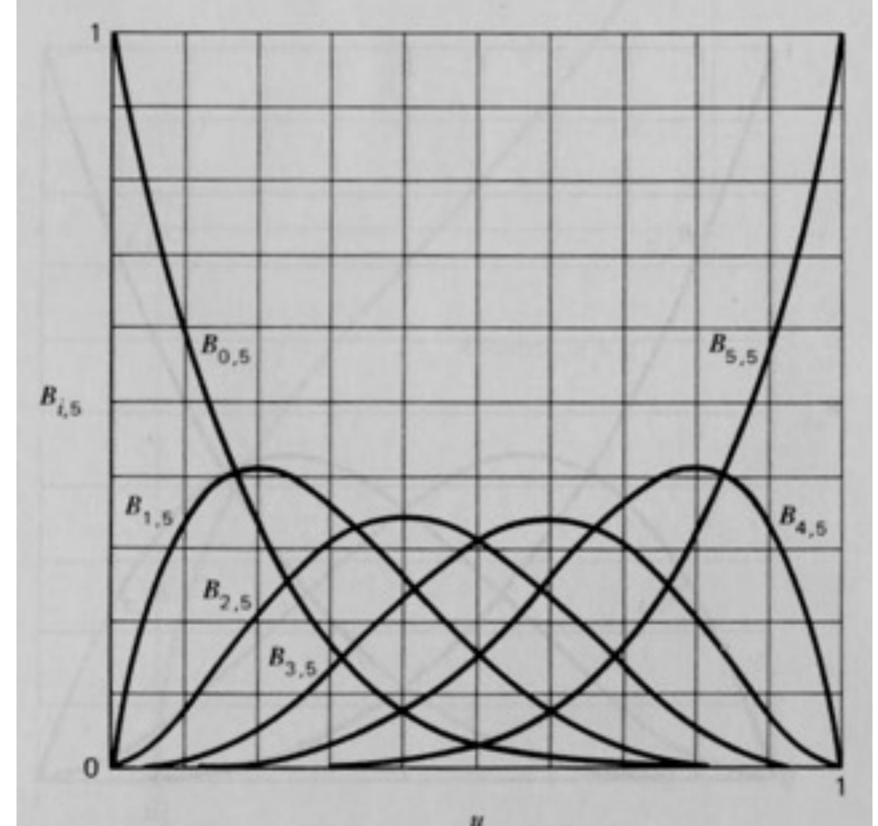
(c)

n = 5

n = 4



(b)

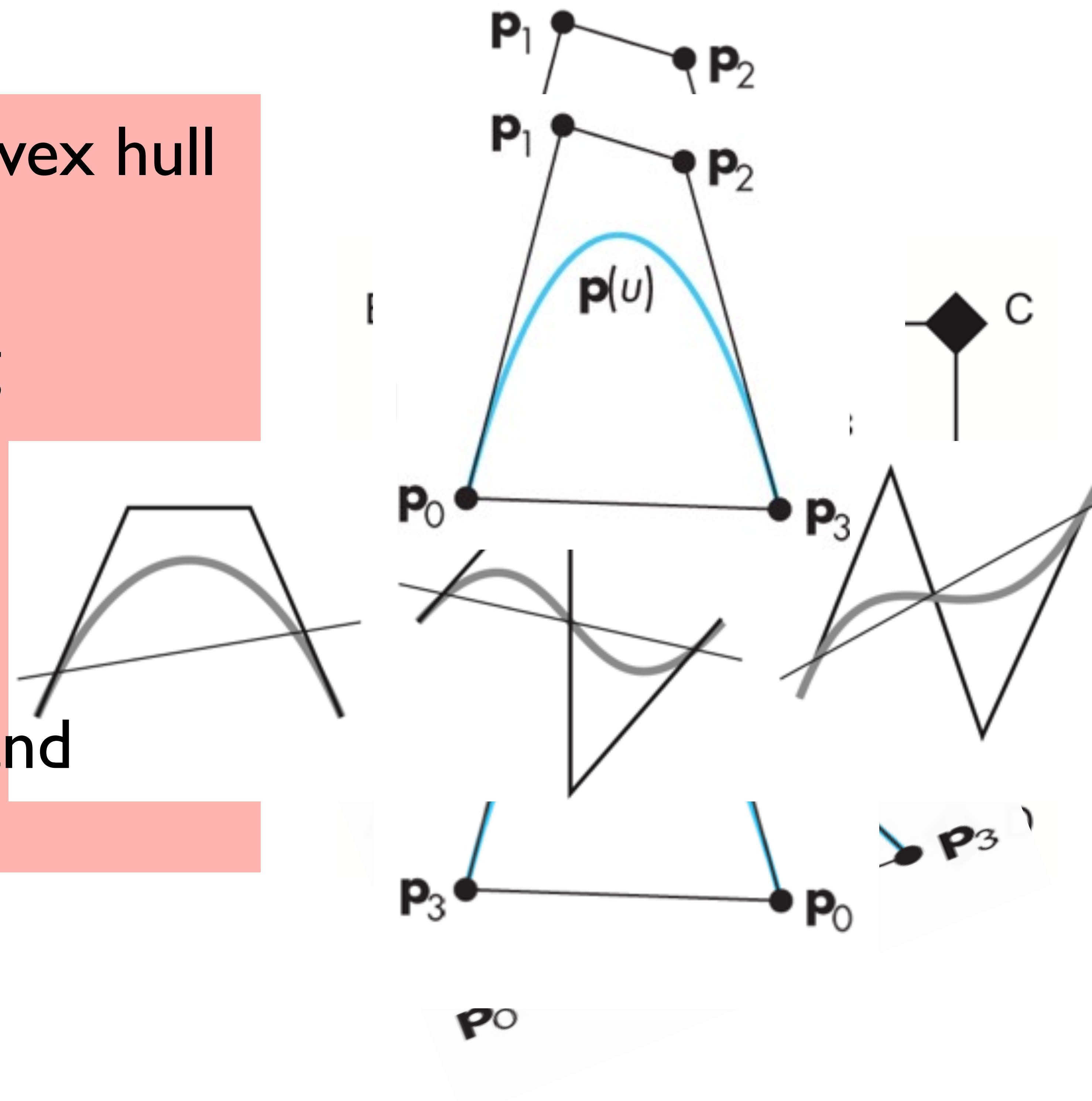


(d)

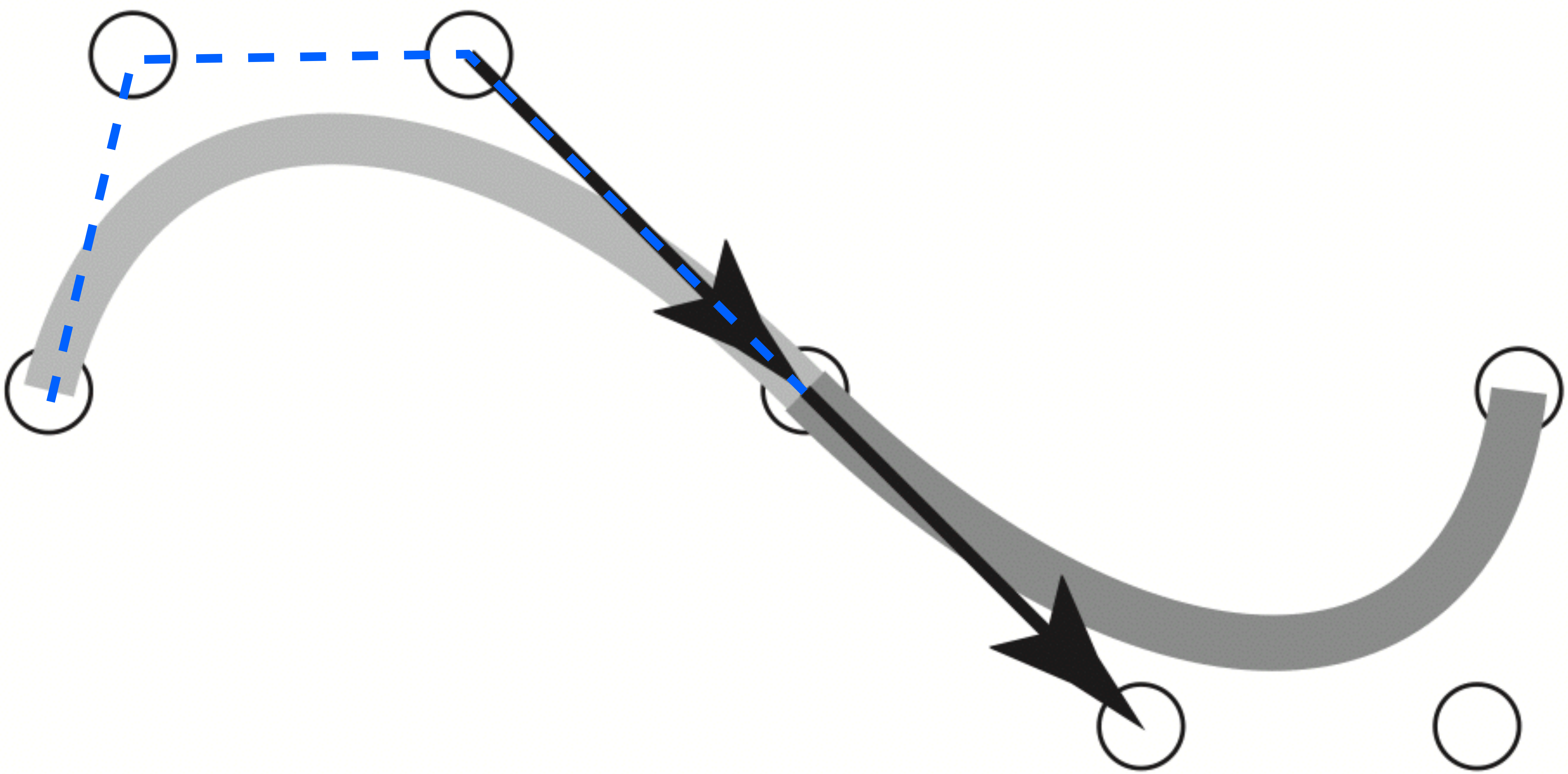
n = 6

Bezier Curve Properties

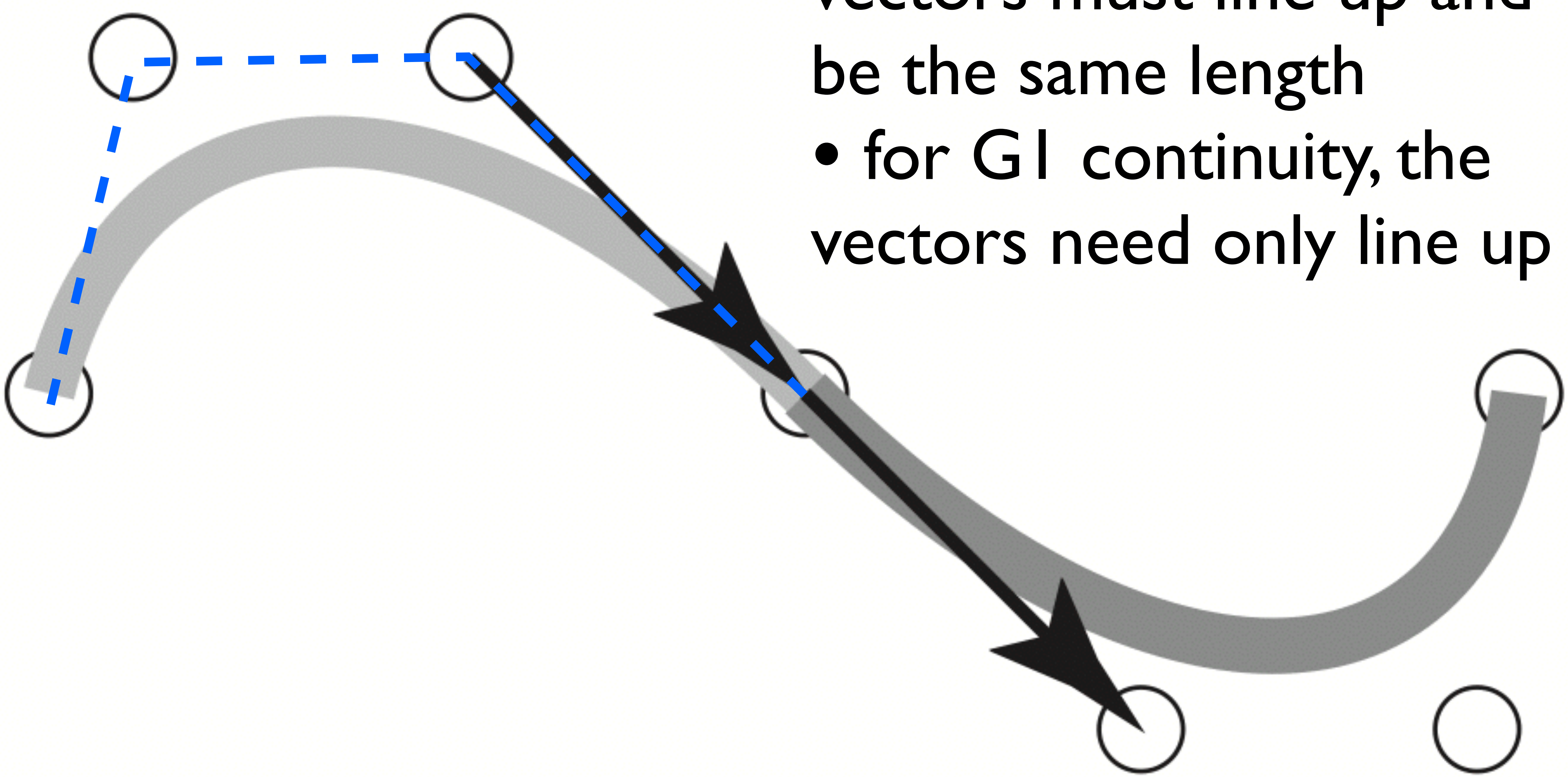
- curve lies in the convex hull of the data
- variation diminishing
- symmetry
- affine invariant
- efficient evaluation and subdivision



Joining Cubic Bezier Curves

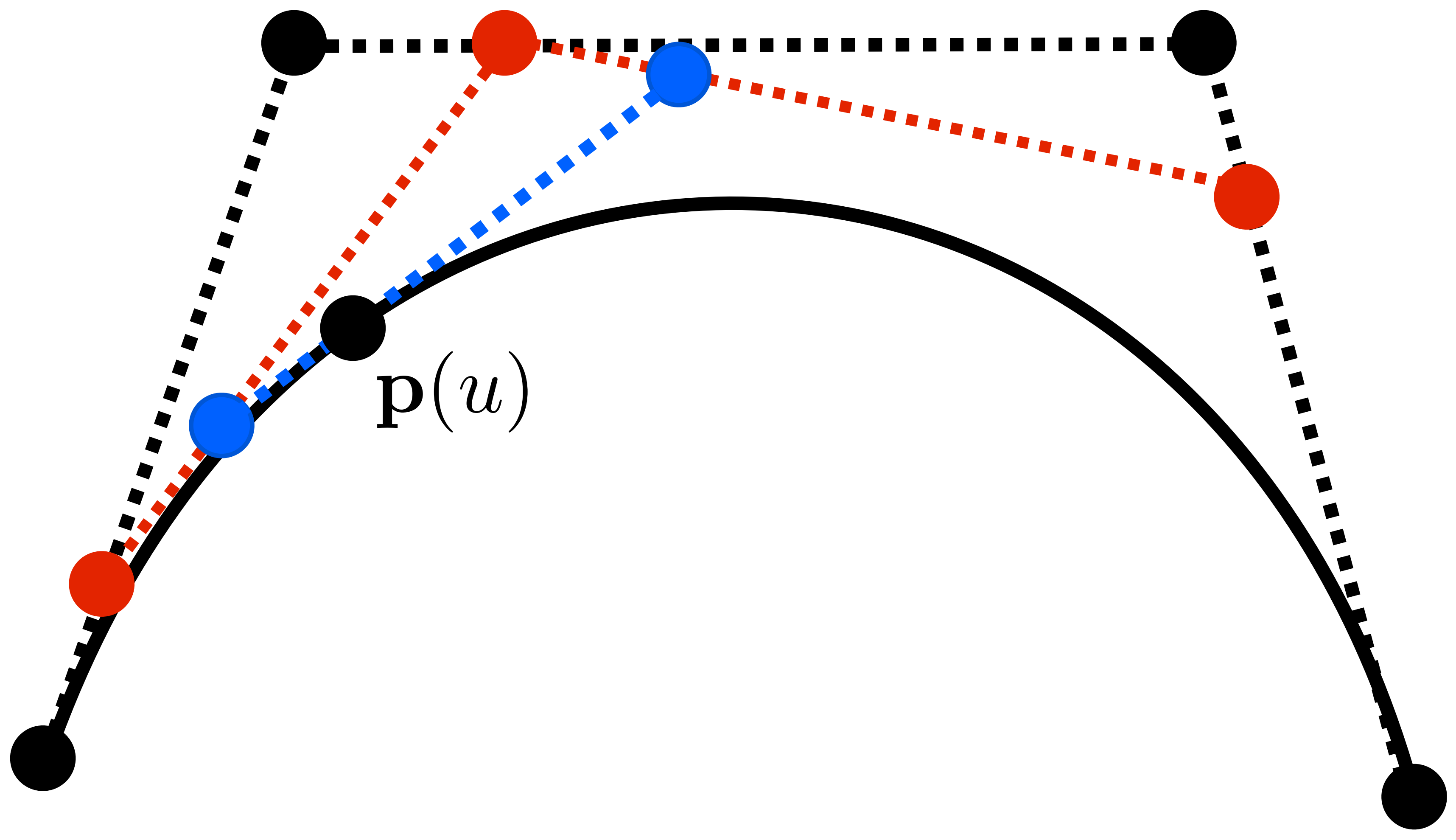


Joining Cubic Bezier Curves

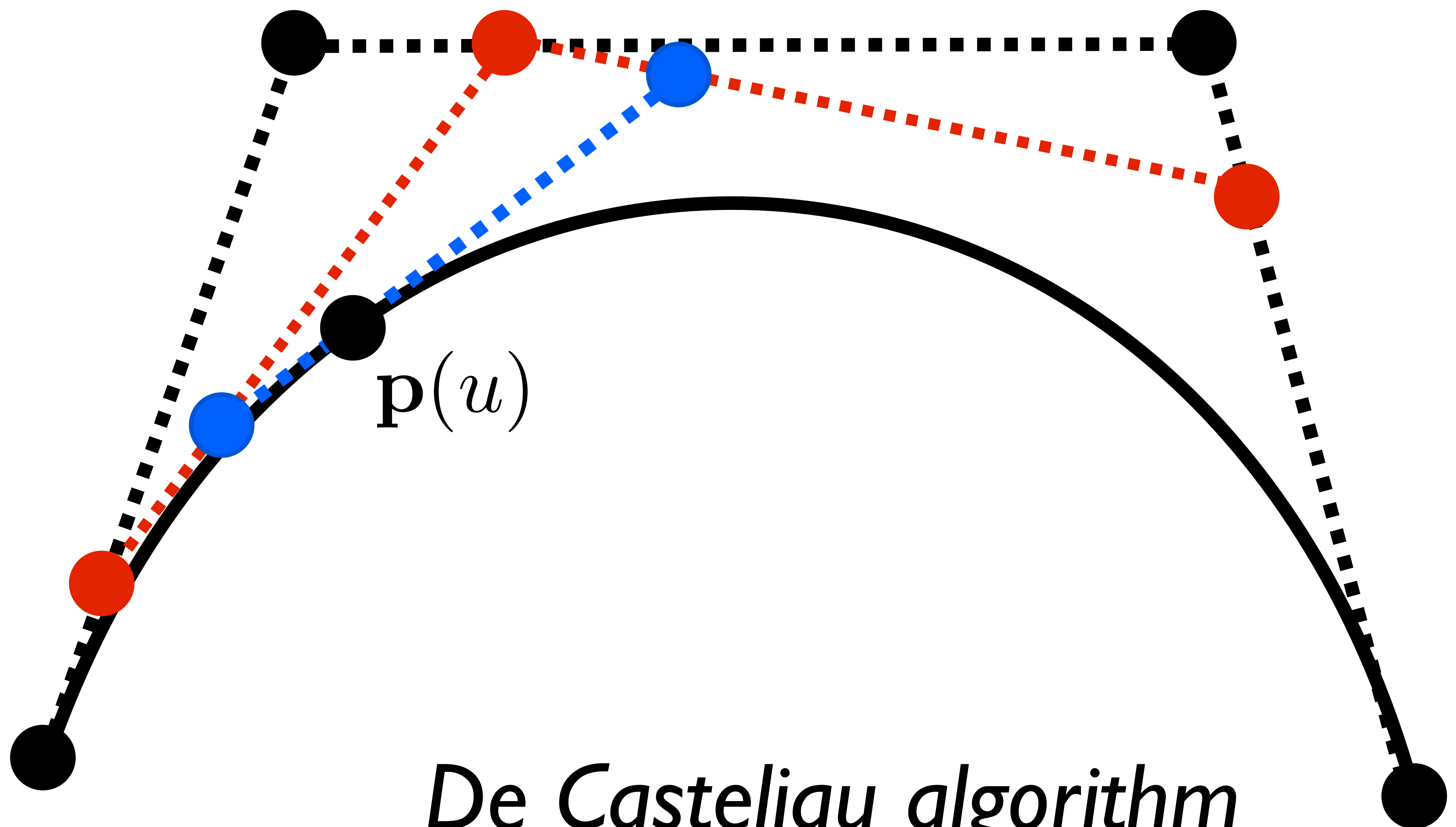


- for C1 continuity, the vectors must line up and be the same length
- for G1 continuity, the vectors need only line up

Evaluating p(u) geometrically

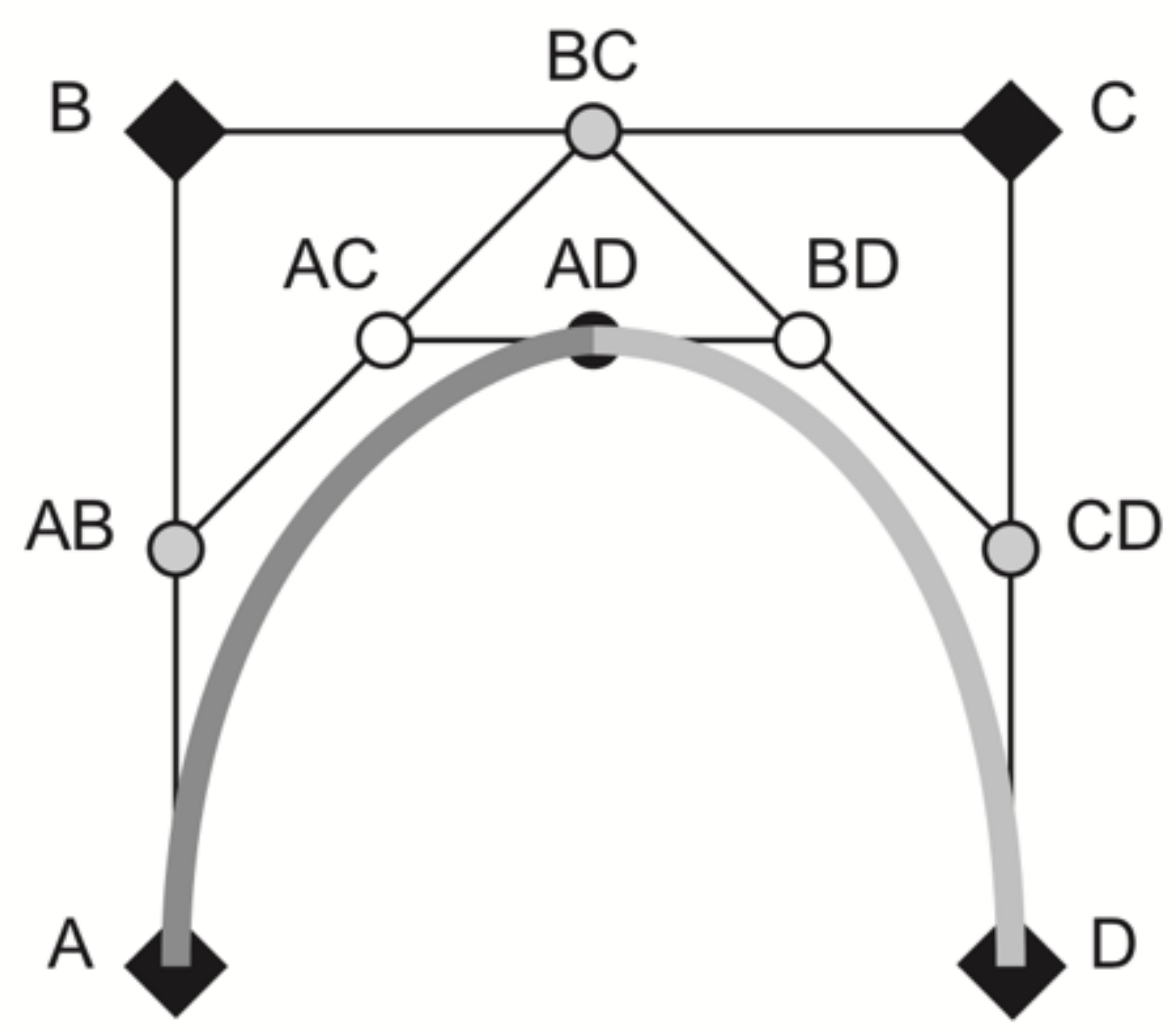


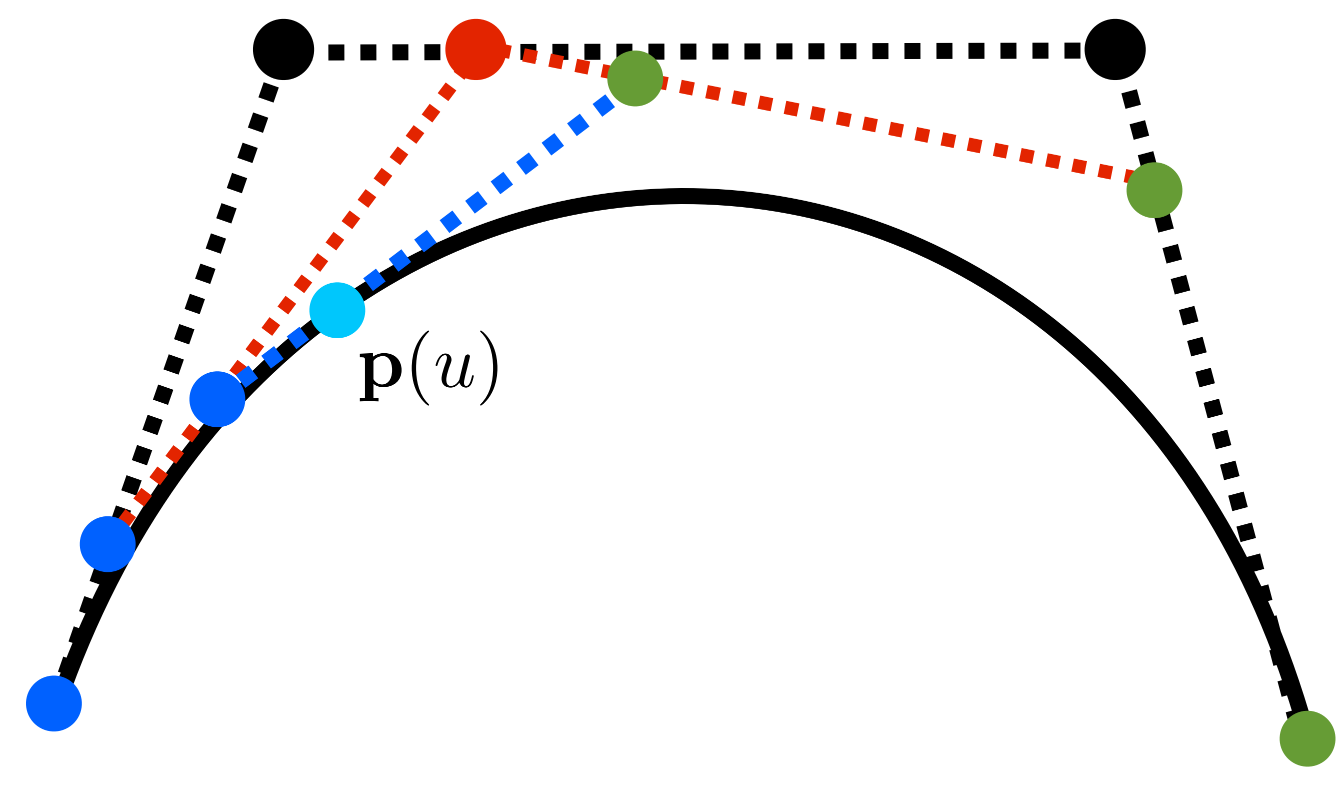
Evaluating p(u) geometrically



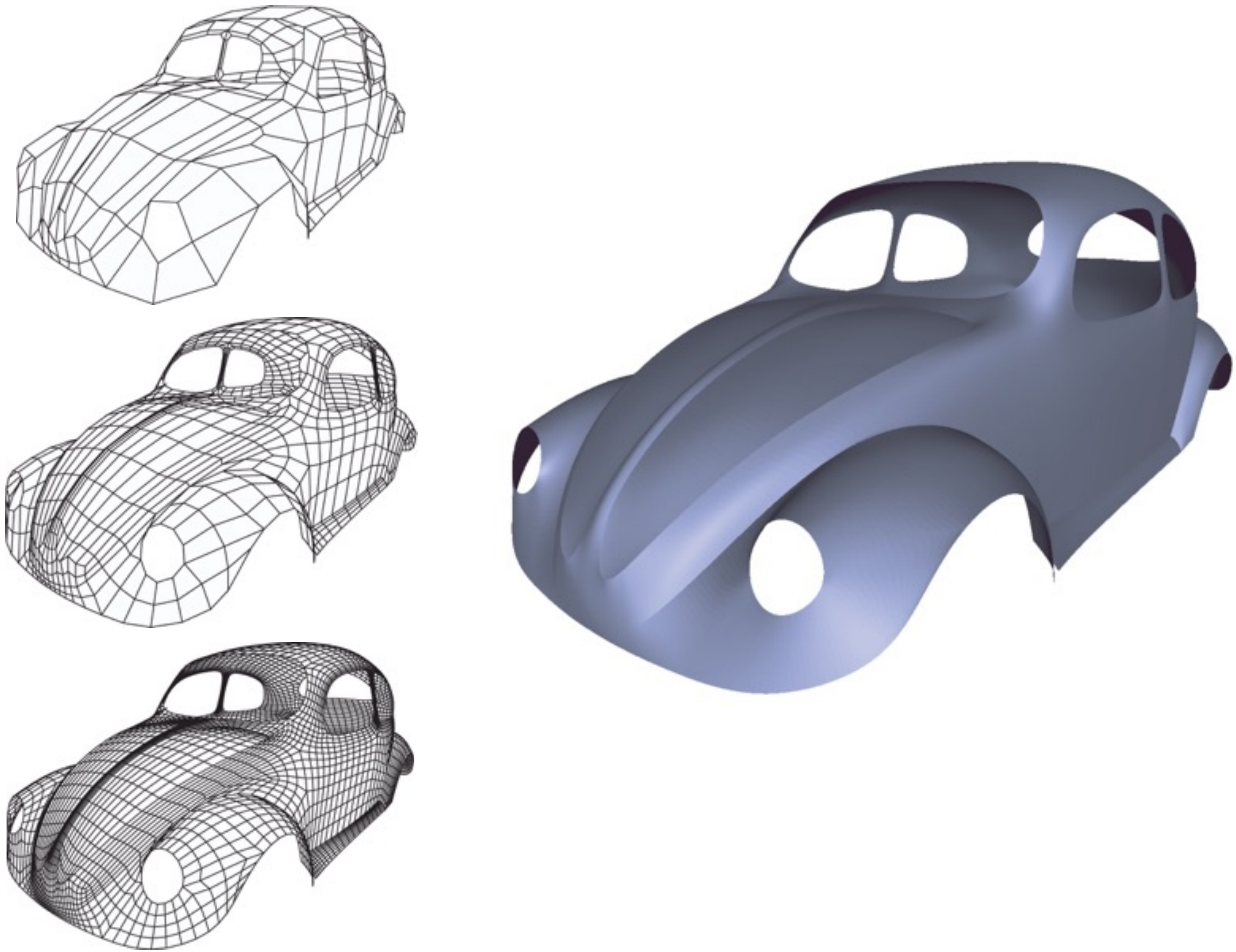
De Casteljau algorithm

Bezier subdivision



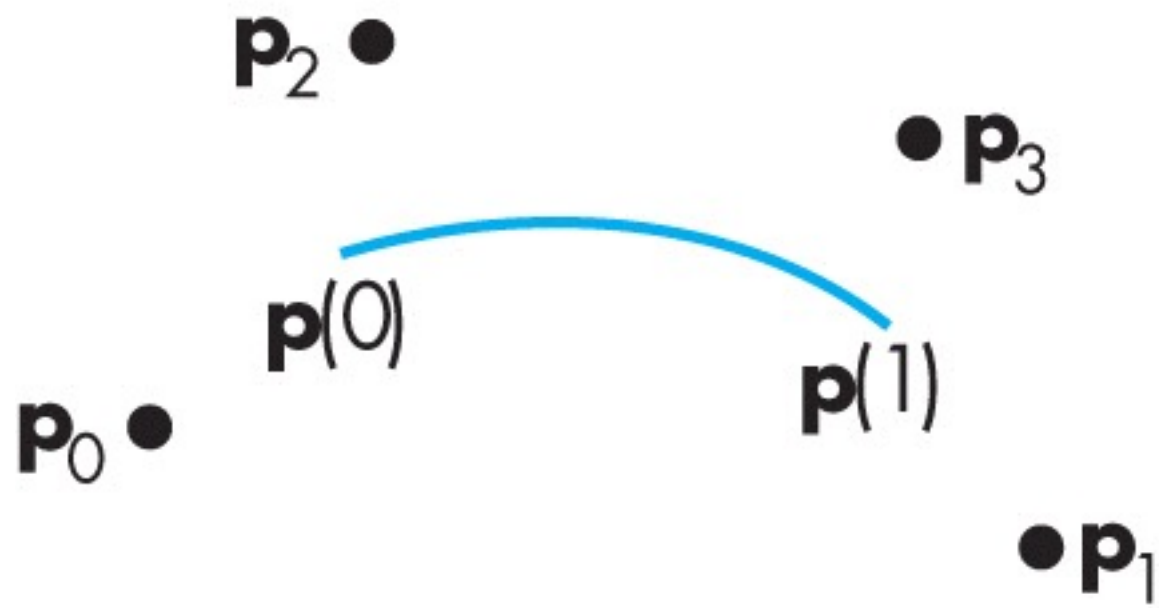


Recursive Subdivision for Rendering



Cubic B-Splines

Cubic B-Splines



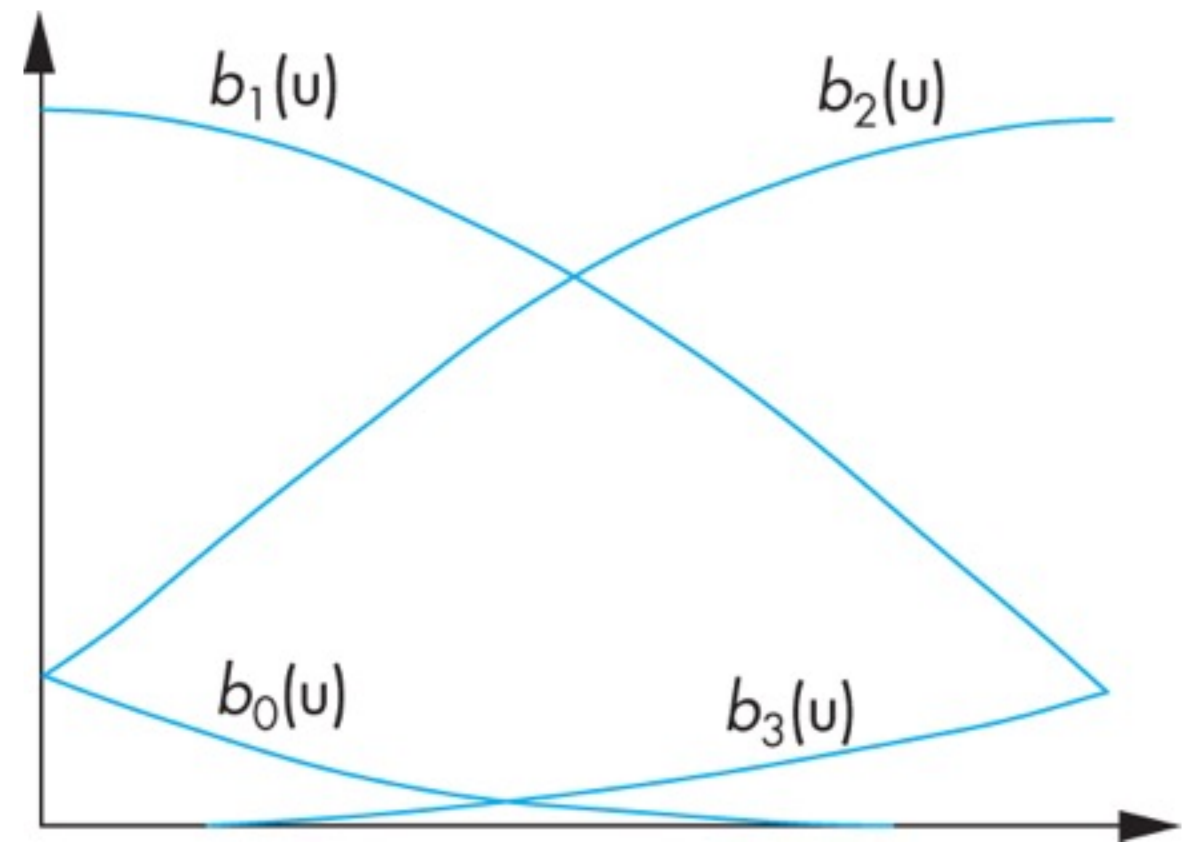
Spline blending functions

$$b_0(u) = \frac{1}{6}(1-u)^3$$

$$b_1(u) = \frac{1}{6}(4 - 6u^2 + 3u^3)$$

$$b_2(u) = \frac{1}{6}(1 + 3u + 3u^2 - 3u^3)$$

$$b_3(u) = \frac{1}{6}u^3$$

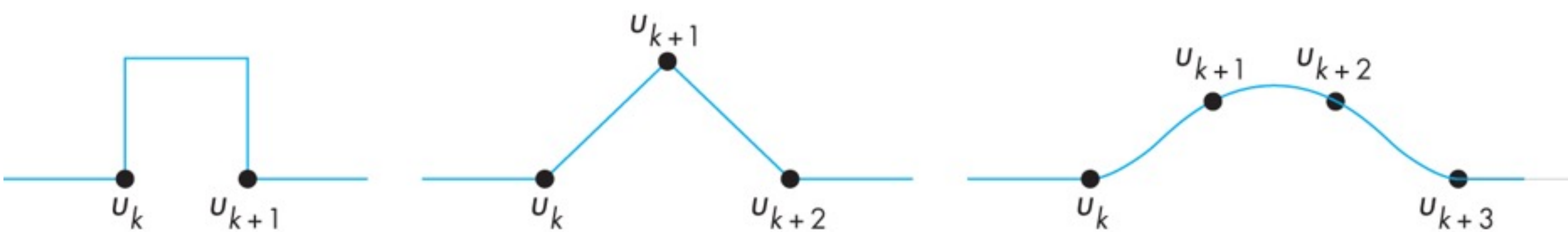


General Splines

- Defined recursively by *Cox-de Boor recursion formula*

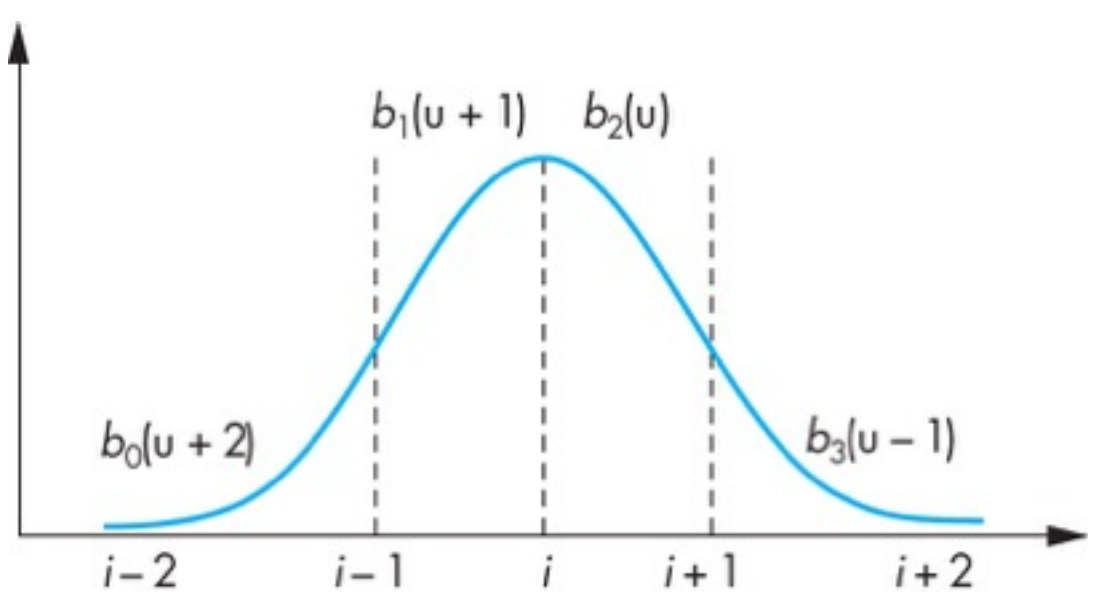
$$b_{j,0}(t) = \begin{cases} 1 & \text{if} \quad t_j \le t \\ 0 & \text{otherwise} \end{cases}$$

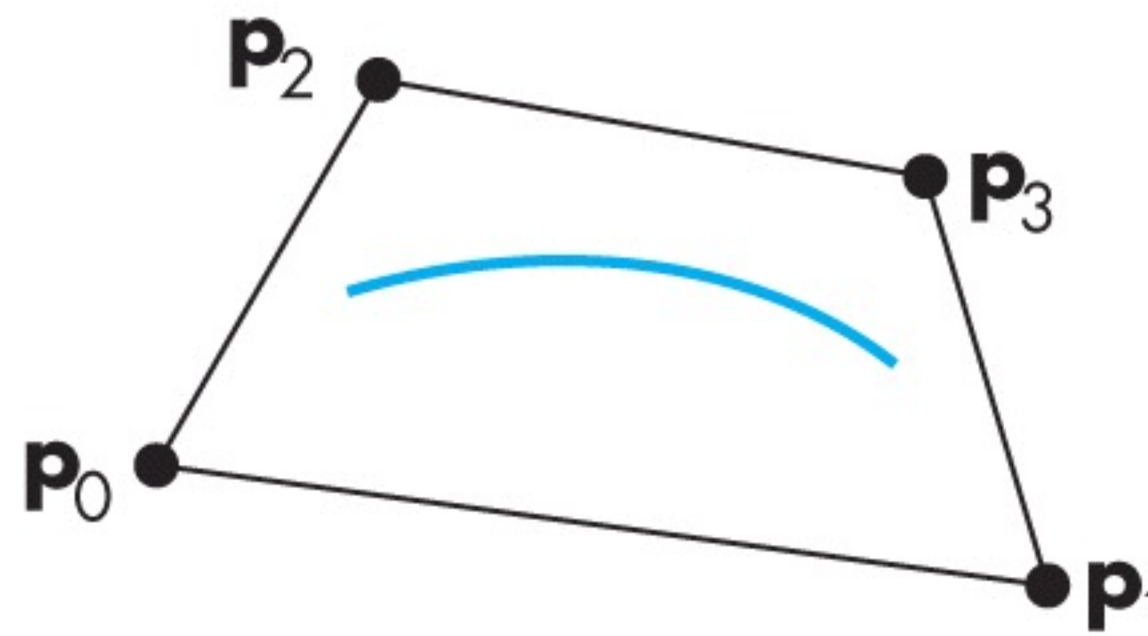
$$b_{j,n}(t) := \frac{t - t_j}{t_{j+n} - t_j} b_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} b_{j+1,n-1}(t)$$



Spline properties

Basis functions



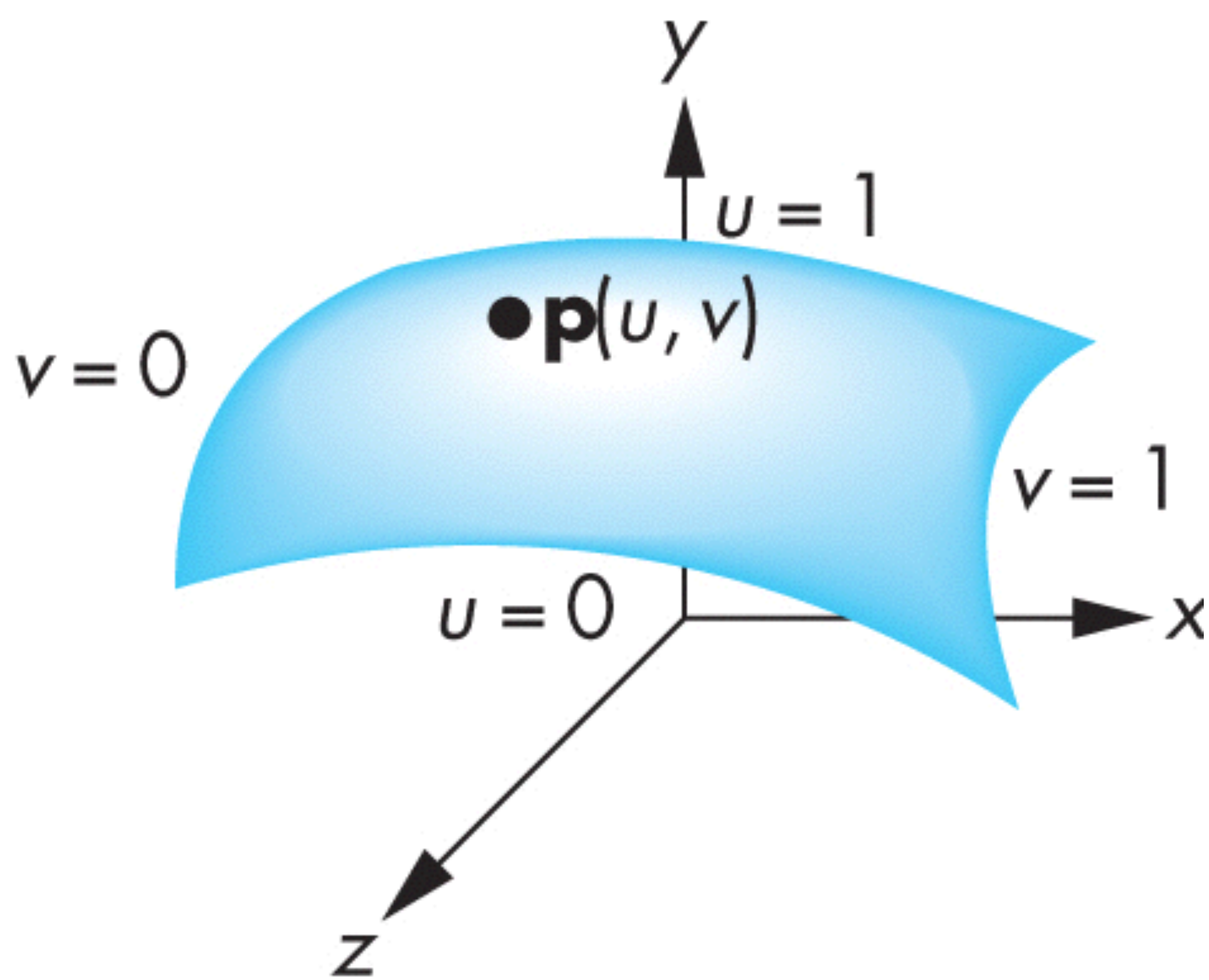


convexity

Surfaces

Parametric Surface

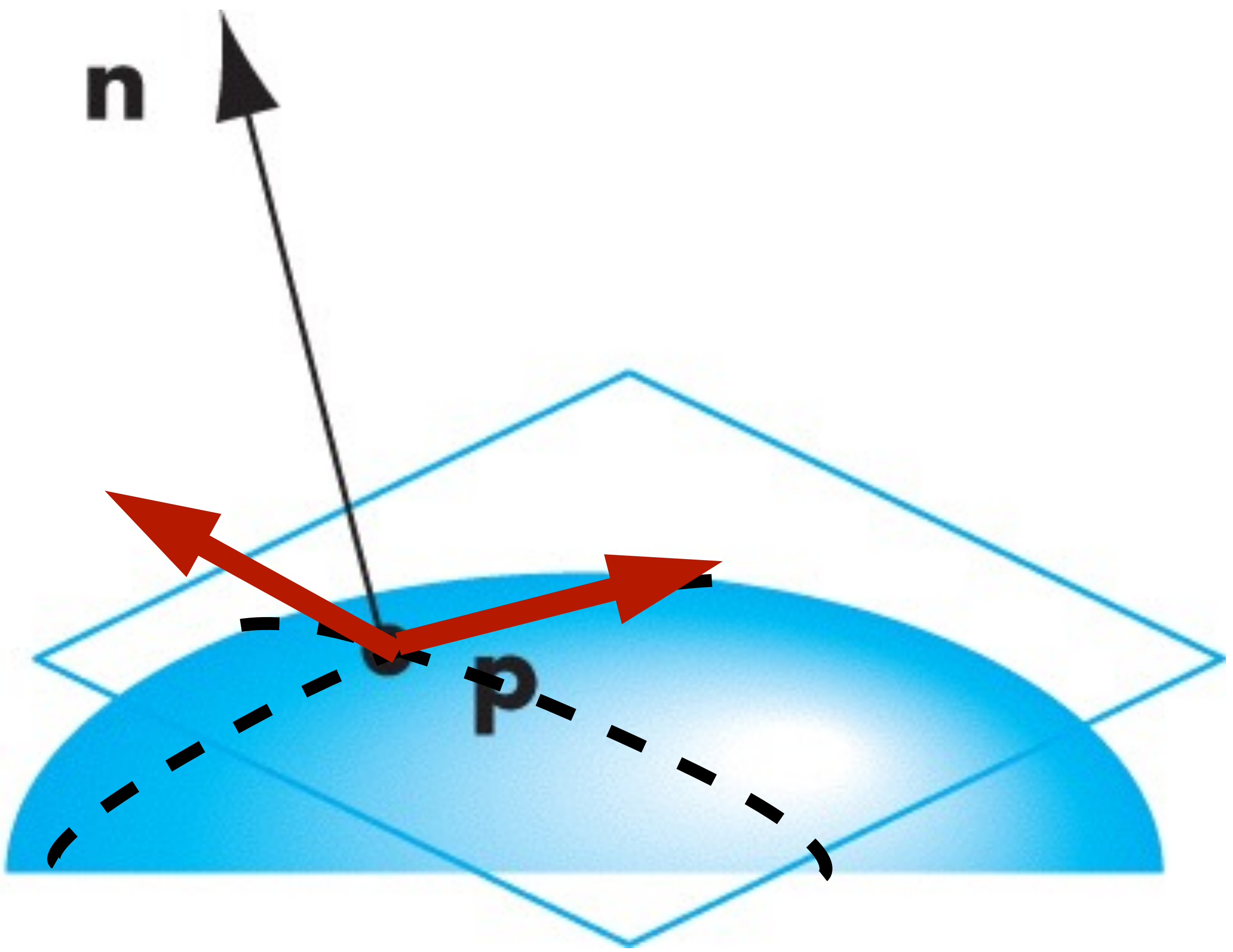
$$x = x(u, v)$$
$$y = y(u, v)$$
$$z = z(u, v)$$



Parametric Surface - tangent plane

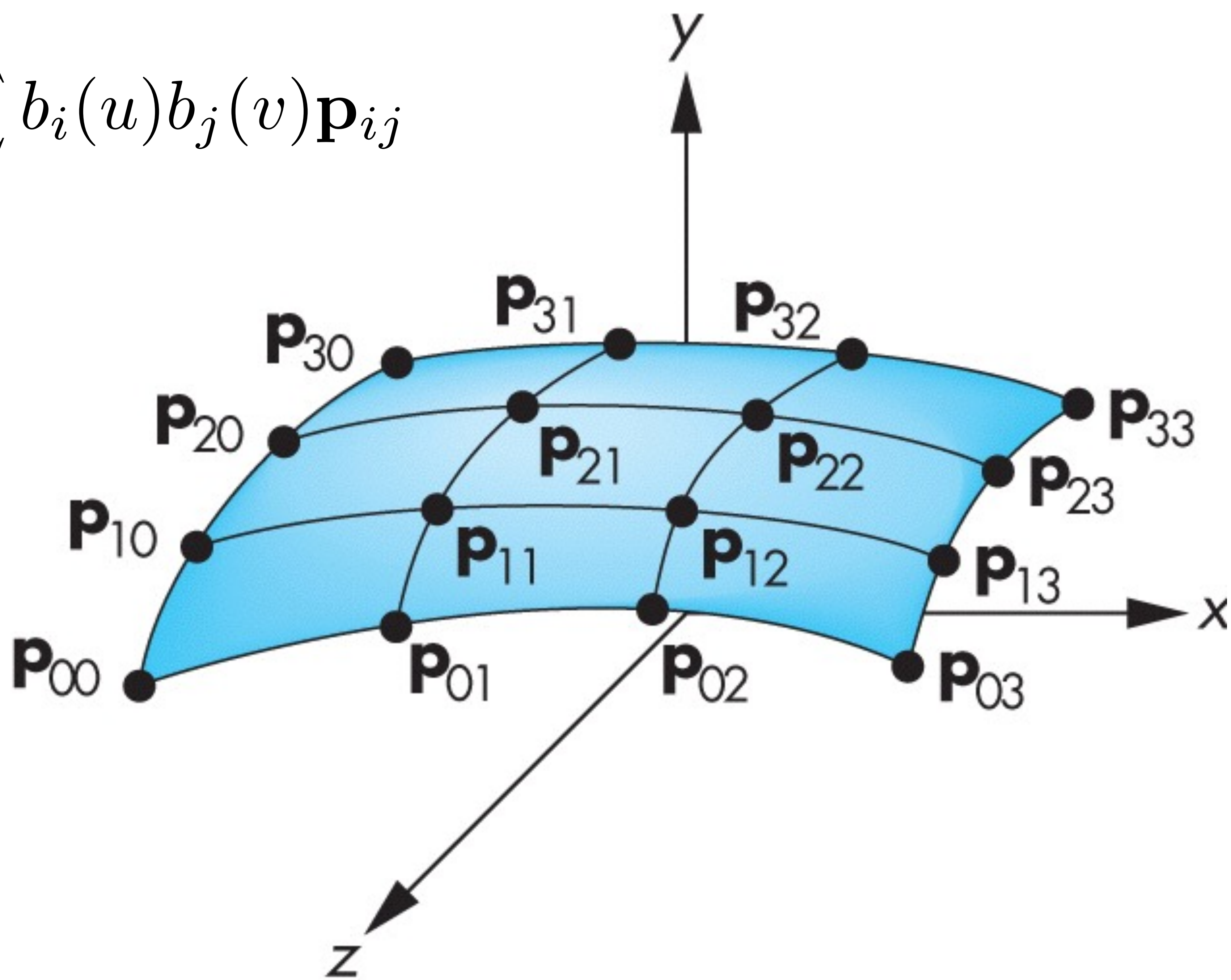
$$\mathbf{t}_u = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}$$

$$\mathbf{t}_v = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}$$



Bicubic Surface Patch

$$\mathbf{f}(u,v) = \sum_i \sum_j b_i(u) b_j(v) \mathbf{p}_{ij}$$



Bezier Surface Patch

$$\mathbf{f}(u,v) = \sum_i \sum_j b_i(u) b_j(v) \mathbf{p}_{ij}$$

Patch lies in convex hull