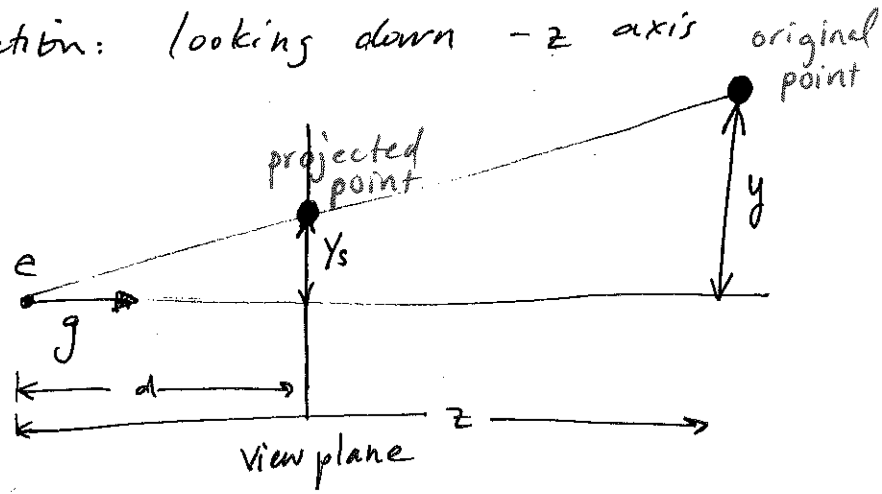
7.2. Projective Transformations.

- Viewpoint at origin
- orientation: looking down $-z$ axis



Using similar triangles,

$$\frac{y_s}{d} = \frac{y}{z} \implies \boxed{y_s = \frac{d}{z} y}$$

1 [Q] How can we represent this with our 4x4 matrices & homogeneous coordinates?

$$y_s = \frac{d}{z} y$$

however, one of the coordinates, z, appears in the denominator

this is not an affine transf.

- Until now, extra coord., has always been 1

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

and has been used to implement translation.

- Also the 4th row of our transf. matrices was

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

preserving the homogeneous coordinate 1.

- Use the 4th component as the denominator of the x-, y-, z- coords; i.e.

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ represents the point $\begin{pmatrix} \frac{x}{w} \\ \frac{y}{w} \\ \frac{z}{w} \end{pmatrix}$

$$\boxed{\text{Note, } w=1 \quad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ is } \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

- Allow other values in the 4th row of our transf. matrix.

<u>linear transformation</u> $\quad ax + by + cz = x'$

<u>affine transformation</u> $\quad ax + by + cz + d = x'$

<u>"linear rational"</u> $\quad \dfrac{ax + by + cz + d}{ex + fy + gz + h} = x'$

[Constraint: all denominators the same]

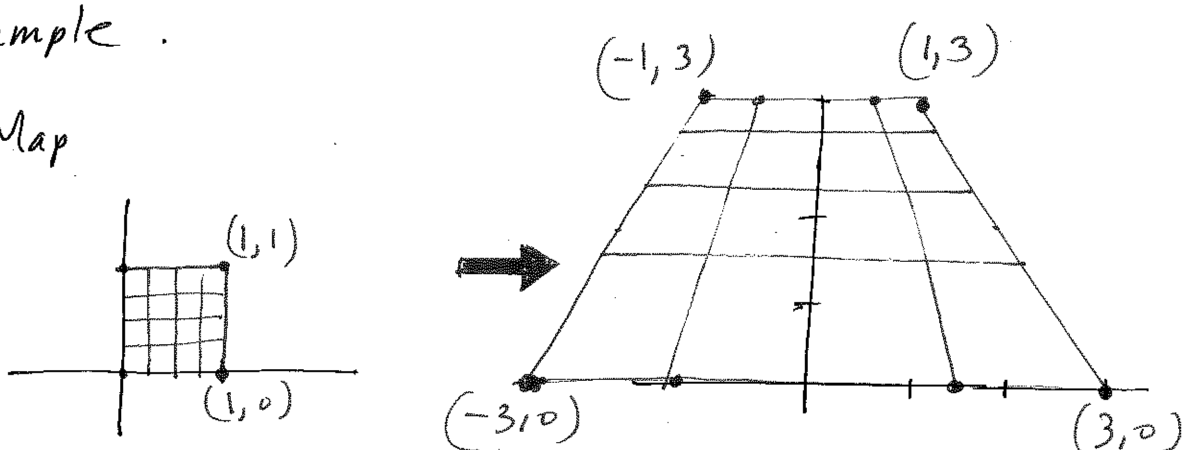
$$\begin{cases} \dfrac{a_1 x + b_1 y + c_1 z + d_1}{ex + fy + gz + h} = x' \\[2ex] \dfrac{a_2 x + b_2 y + c_2 z + d_2}{ex + fy + gz + h} = y' \\[2ex] \dfrac{a_3 x + b_3 y + c_3 z + d_3}{ex + fy + gz + h} = z' \end{cases}$$

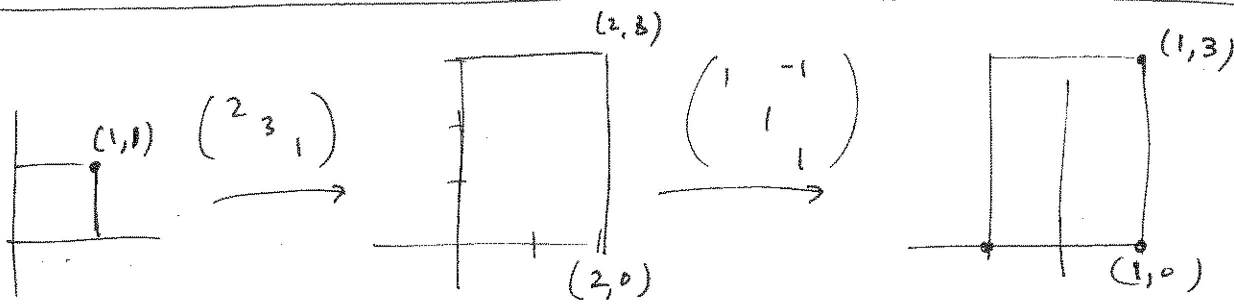
In matrix form,

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\\ \tilde{w}\end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\\ e & f & g & h\end{pmatrix}\begin{pmatrix}x\\ y\\ z\\ 1\end{pmatrix} \Rightarrow \begin{cases} x' = \dfrac{\tilde{x}}{\tilde{w}}\\ y' = \dfrac{\tilde{y}}{\tilde{w}}\\ z' = \dfrac{\tilde{z}}{\tilde{w}}\end{cases}$$

Example.

Map



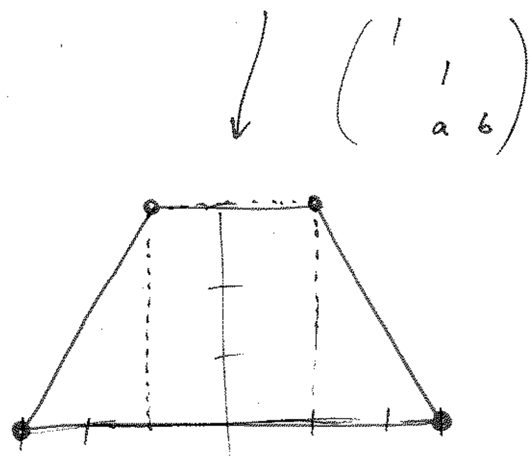


$$\begin{cases} x' = \dfrac{x}{ay+b}\\ y' = \dfrac{y}{ay+b}\end{cases}$$

Find a, b s.t.

$(1,0) \to (3,0)$

$(1,3) \to (1,3)$



$(1,0) \to (3,0)$

$$x' = \frac{1}{a\cdot 0+b} = \frac{1}{b} \Rightarrow \boxed{b=\frac{1}{3}}$$

$$y' = \frac{0}{a\cdot 0+b} = 0 \checkmark$$

$(1,3) \to (1,3)$

$$x' = \frac{1}{a\cdot 3+\frac{1}{3}} = \frac{1}{\frac{9a+1}{3}} = \frac{3}{9a+1} = 1 \Rightarrow 3 = 9a+1 \Rightarrow \boxed{a=\frac{2}{9}}$$

$$y' = \frac{3}{a\cdot 3+\frac{1}{3}} = \frac{3}{\frac{2}{3}+\frac{1}{3}} = 3 \checkmark$$

Notes: y not scaled uniformly

$$y' = \frac{y}{ay+b}$$

Perspective Example

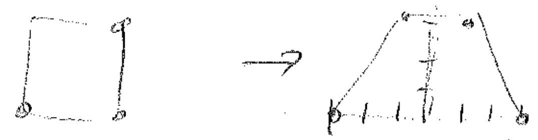
LECTURE 6

$$M = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$M\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\\frac{1}{3}\end{pmatrix} \quad \Big| \quad M\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\\frac{1}{3}\end{pmatrix} \quad \Big| \quad M\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix} \quad \Big| \quad M\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\1\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}0\\0\end{pmatrix} \to \begin{pmatrix}-3\\0\end{pmatrix} \qquad \begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}3\\0\end{pmatrix} \qquad \begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}1\\3\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-1\\3\end{pmatrix}$$



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$$\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} dx \\ dy \\ dz \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{d}{z}x \\ \frac{d}{z}y \\ d \\ 1 \end{pmatrix}$$

change this equation

$$\frac{az+b}{z} = a + \frac{b}{z}$$

$$\boxed{P = \begin{pmatrix} n & & & \\ & n & & \\ & & n+f & -nf \\ & & 1 & 0 \end{pmatrix}}$$

criteria

1. $n \to n \qquad a + \frac{b}{n} = n$
2. $f \to f \qquad a + \frac{b}{f} = f$

$$\frac{b}{n} - \frac{b}{f} = n - f$$
$$fb - nb = nf(n-f)$$
$$(f-n)b = nf(n-f)$$
$$b = -nf$$
$$\Rightarrow a = n + \frac{nf}{n} = n + f$$

$$\Rightarrow \boxed{\begin{aligned} a &= n+f \\ b &= -nf \end{aligned}}$$

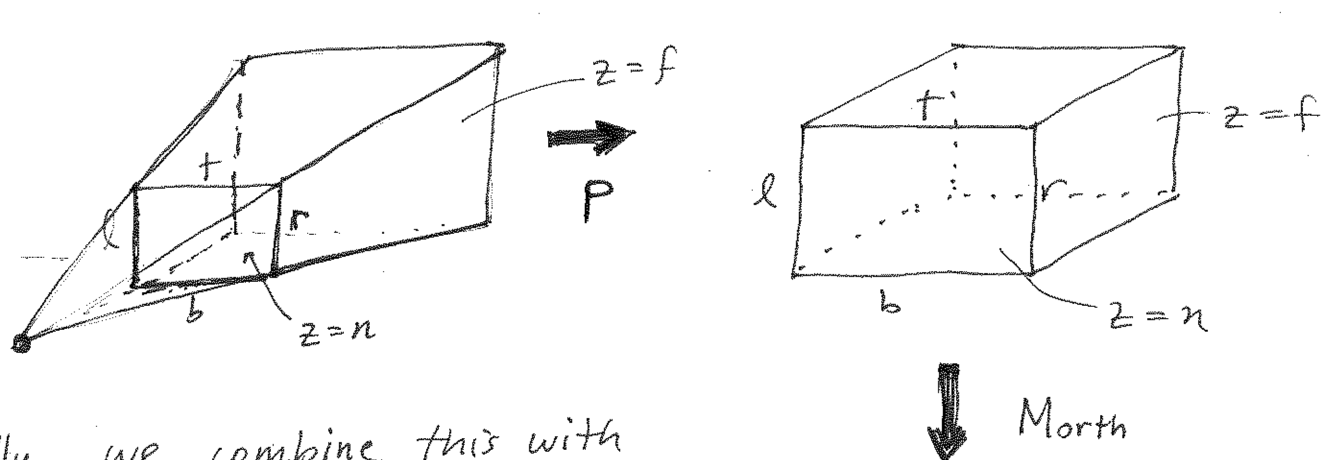
- substituting these results into our transformation, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{n+f}{n} & -f \\ 0 & 0 & 1/n & 0 \end{bmatrix}$$

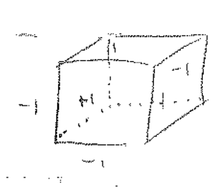
- to avoid the division, we multiply through by n (recalling that $\alpha(x,y,z,w)^T$ & $(x,y,z,w)^T$ give the same point)

$$P = \begin{bmatrix} n & & & \\ & n & & \\ & & n+f & -nf \\ & & 1 & 0 \end{bmatrix}$$

- this has achieved the transformation



- Finally, we combine this with the orthographic transformation, M_{orth}, to get to the canonical viewing volume, so $\boxed{M_{per} = M_{orth} P}$



recall that

$$M_{orth} = \begin{bmatrix} \frac{2}{r-l} & & & -\frac{(l+r)}{(r-l)} \\ & \frac{2}{t-b} & & \frac{-(b+t)}{t-b} \\ & & \frac{2}{n-f} & -\frac{(f+n)}{n-f} \\ & & & 1 \end{bmatrix}$$

- so

$$M_{per} = M_{orth} P = \begin{bmatrix} \frac{2n}{r-l} & & \frac{-(l+r)}{r-l} & \\ & \frac{2n}{t-b} & \frac{-(b+t)}{t-b} & \\ & & \frac{n+f}{n-f} & -\frac{2nf}{n-f} \\ & & 1 & 0 \end{bmatrix}$$

- Note : OpenGL ① assumes a right-handed system <u>before</u> proj. and a left-handed system <u>after</u> proj.
  ② assumes that glOrtho and glFrustum will be passed near = $-n$ and far = $-f$

① multiple 3rd row of matrix by -1

② replace n and f in the above with $-|n|$ and $-|f|$

also mult. whole matrix by (-1)

$$M^{openGL}_{orth} = \begin{bmatrix} \frac{2}{r-l} & & & \frac{-(l+r)}{r-l} \\ & \frac{2}{t-b} & & \frac{-(b+t)}{t-b} \\ & & \frac{2}{|n|-|f|} & \frac{|f|+|n|}{|n|-|f|} \\ & & & 1 \end{bmatrix} ; \quad M^{openGL}_{per} = \begin{bmatrix} \frac{2|n|}{r-l} & & \frac{l+r}{r-l} & \\ & \frac{2|n|}{t-b} & \frac{b+t}{t-b} & \\ & & \frac{|n|+|f|}{|n|-|f|} & \frac{2|n||f|}{|n|-|f|} \\ & & -1 & \end{bmatrix}$$