CS130 : Computer Graphics

Curves (cont.)

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Blending Functions

Blending functions are more convenient basis than monomial basis

- "canonical form" (monomial basis)

$$\mathbf{f}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3$$

- "geometric form" (blending functions)

$$\mathbf{f}(u) = b_0(u)\mathbf{p}_0 + b_1(u)\mathbf{p}_1 + b_2(u)\mathbf{p}_2 + b_3(u)\mathbf{p}_3$$

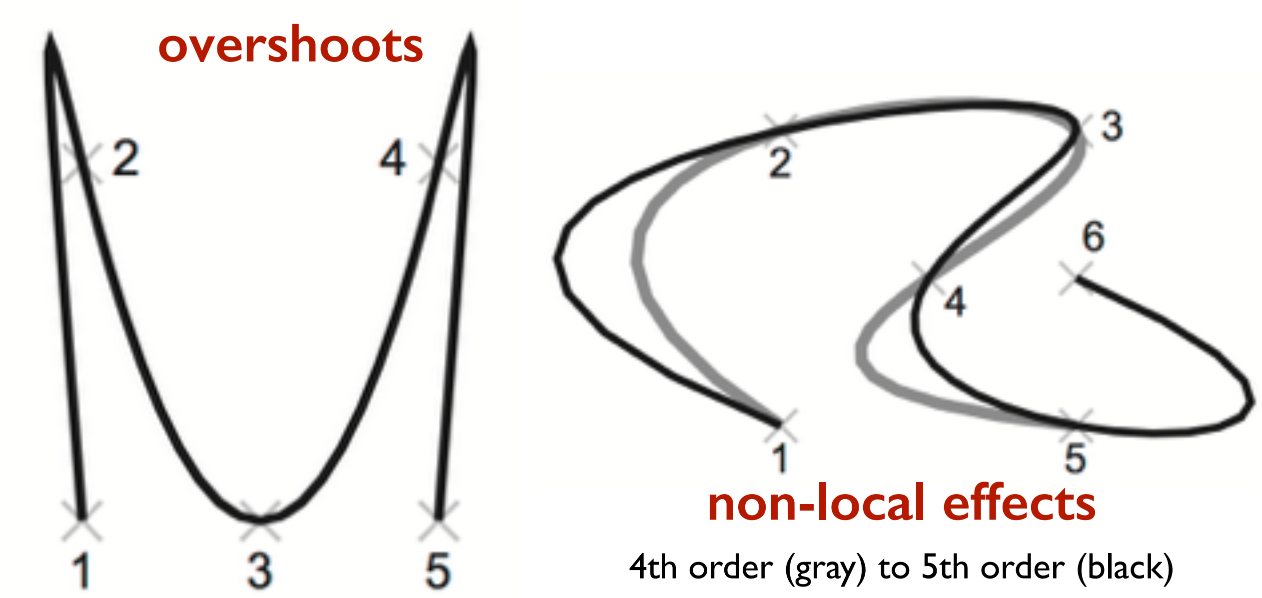
- geometric form (bottom) is more intuitive because it combines control points with blending functions
[see Shirley Section 15.3]

Interpolating Polynomials

Interpolating polynomials

- Given n+1 data points, can find a unique interpolating polynomial of degree n
- Different methods:
  - Vandermonde matrix
  - Lagrange interpolation
  - Newton interpolation

higher order interpolating polynomials are rarely used

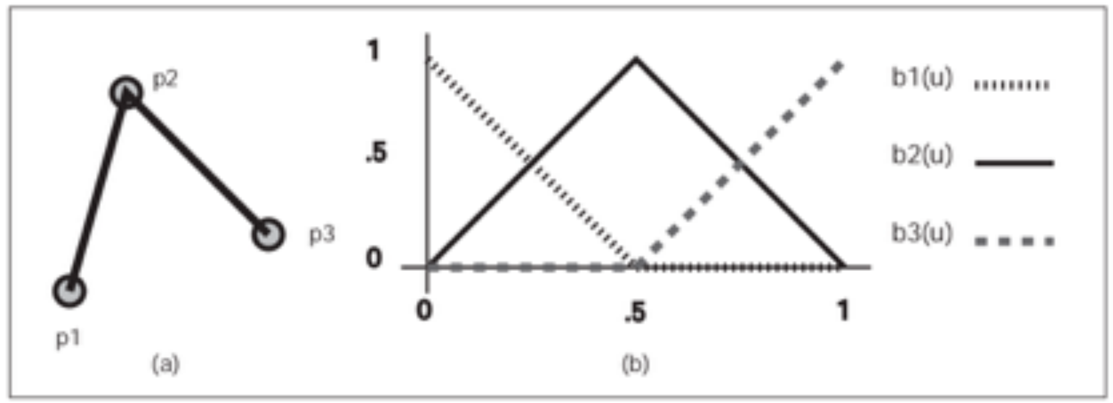


These images demonstrate problems with using higher order polynomials:
- overshoots
- non-local effects (in going from the 4th order polynomial in grey to the 5th order polynomial in black)

Piecewise Polynomial Curves

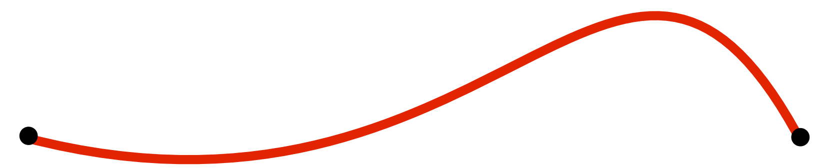
Example: blending functions for two line segments

$$\mathbf{f}(u) = \begin{cases} \mathbf{f}_1(2u) & u \leq 0.5 \\ \mathbf{f}_2(2u-1) & u > 0.5 \end{cases}$$



b1(u) = 1-2u, 0 <= u <= .5
0 .5 <= u <= 1
b2(u) = 2u, 0 <= u <= .5
2(1-u), .5 <= u <= 1
b3(u) = 0 , 0 <= u <= .5
2u-1, .5 <= u <= 1

Cubics



$$\mathbf{f}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3$$

- Allow up to C^2 continuity at knots
- need 4 control points
  - may be 4 points on the curve, combination of points and derivatives, ...
- good smoothness and computational properties

need 4 control points: might be 4 points on the curve, combination of points and derivatives, ...

We can get any 3 of 4 properties

1. piecewise cubic
2. curve interpolates control points
3. curve has local control
4. curves has C2 continuity at knots

Cubics

- Natural cubics
  - C2 continuity
  - n points -> n-1 cubic segments
- control is non-local :(
- ill-conditioned x(

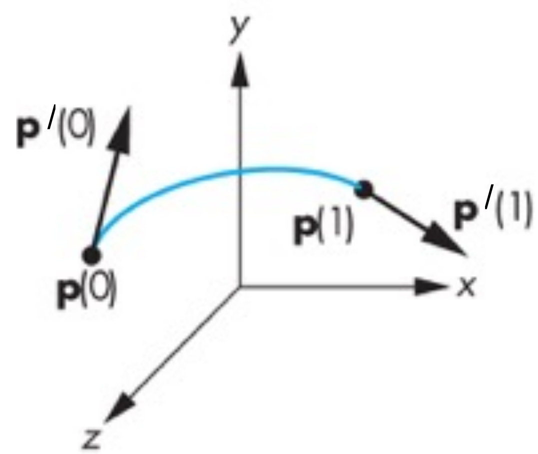
Cubic Hermite Curves

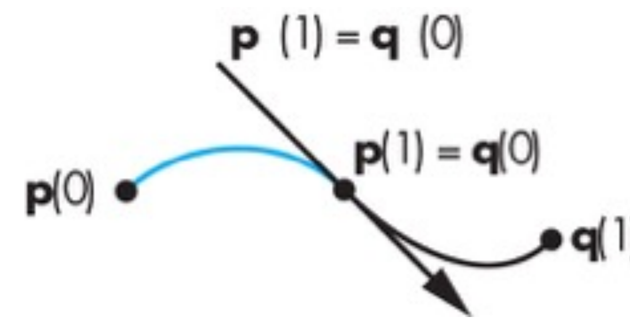
- C1 continuity
- specify both positions and derivatives

Cubic Hermite Curves

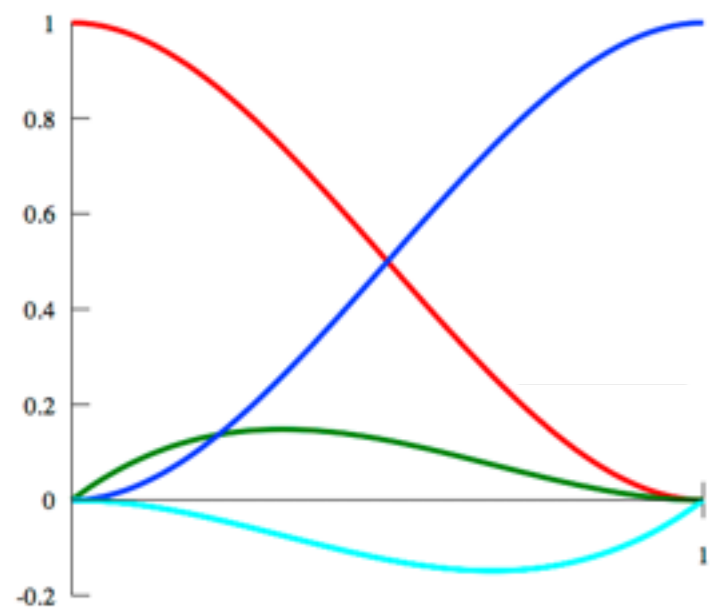
Specify endpoints and derivatives

construct curve with C^1 continuity





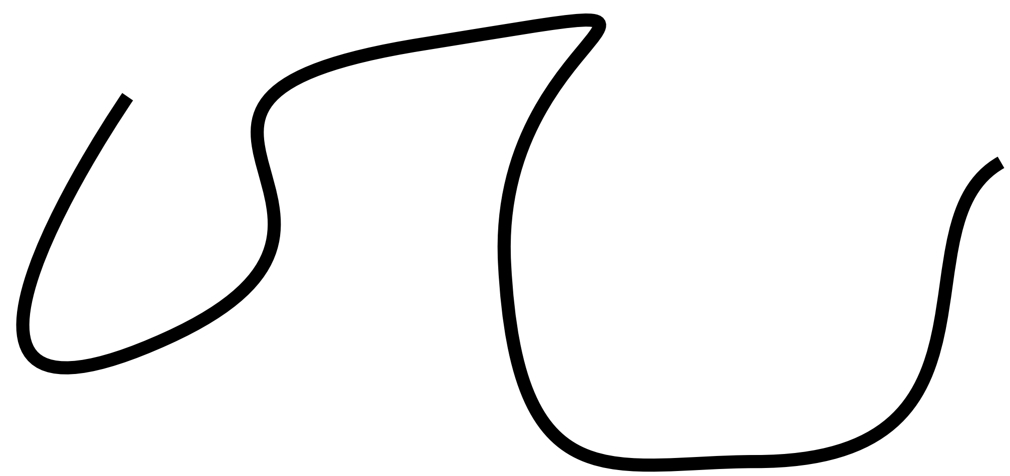
Hermite blending functions



[Wikimedia Commons]

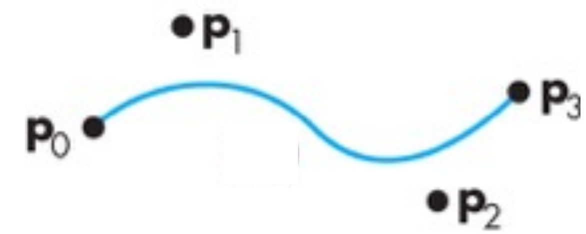
$$b_0(u) = 2u^3 - 3u^2 + 1$$
$$b_1(u) = -2u^3 + 3u^2$$
$$b_2(u) = u^3 - 2u^2 + u$$
$$b_3(u) = u^3 - u^2$$

Example: keynote curve tool



Interpolating vs. Approximating Curves

Interpolating

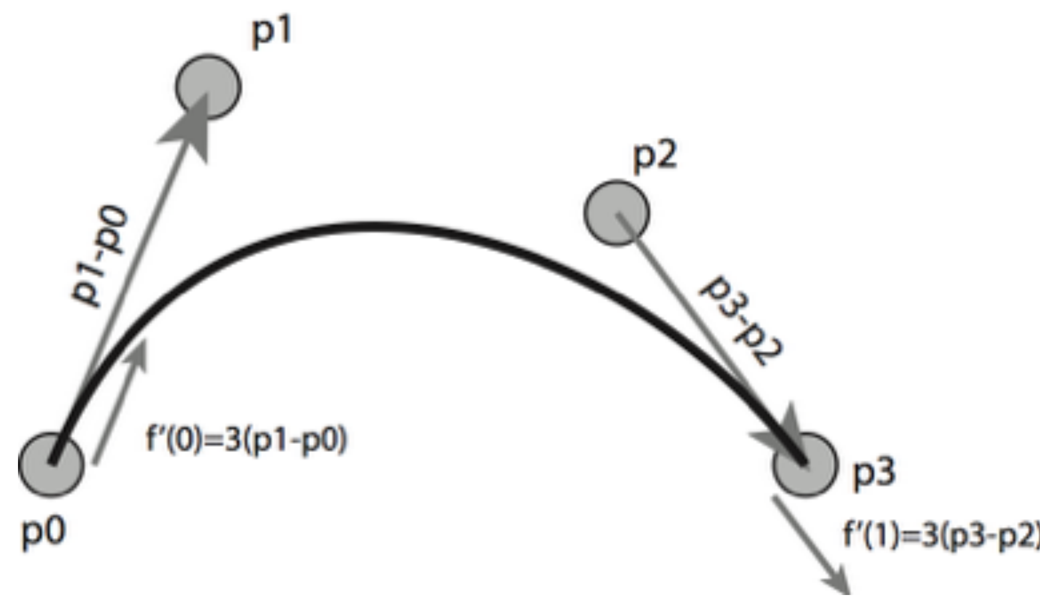


Approximating
(non-interpolating)

approximating

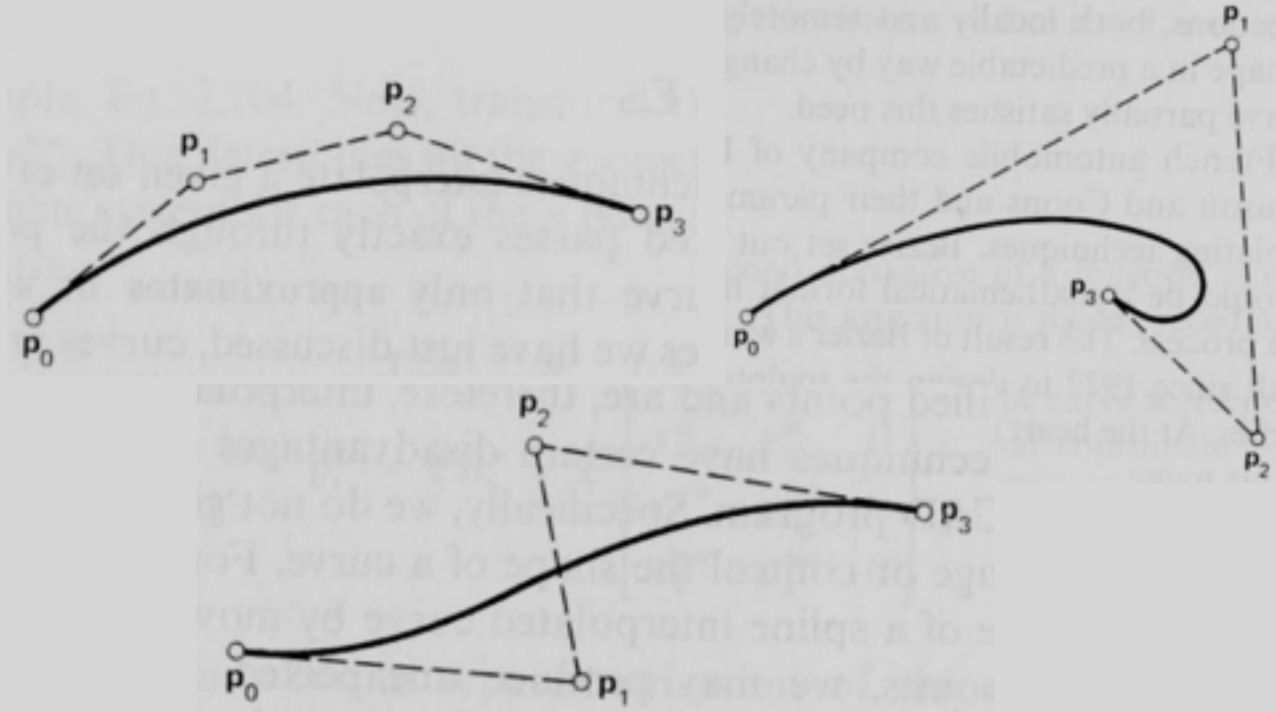
Cubic Bezier Curves

Cubic Bezier Curves

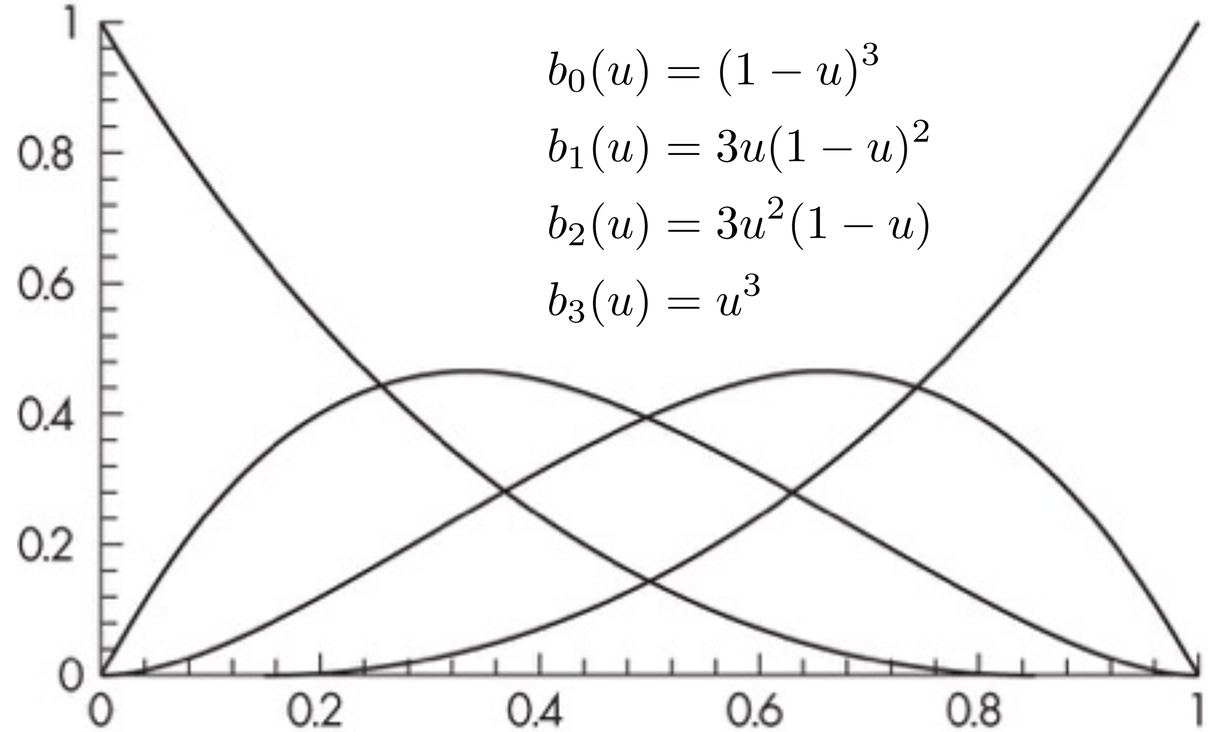


-The curve interpolates its first (u=0) and last (u = 1) control points
- first derivative at the beginning is the vector from first to second point, scaled by degree

Cubic Bezier Curve Examples

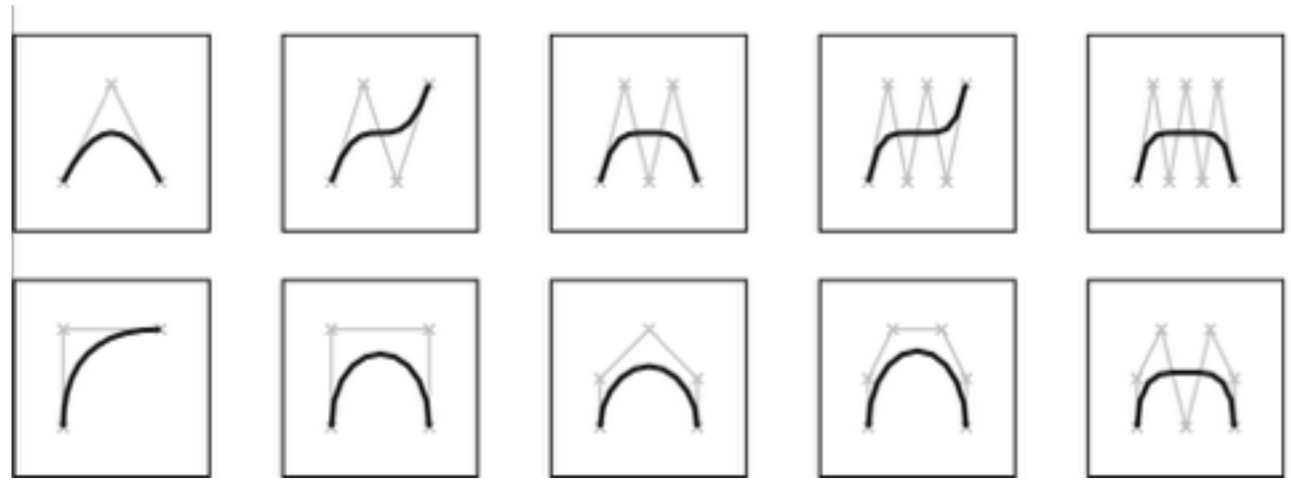


Cubic Bezier blending functions



$$b_0(u) = (1-u)^3$$
$$b_1(u) = 3u(1-u)^2$$
$$b_2(u) = 3u^2(1-u)$$
$$b_3(u) = u^3$$

Bezier Curves Degrees 2-6

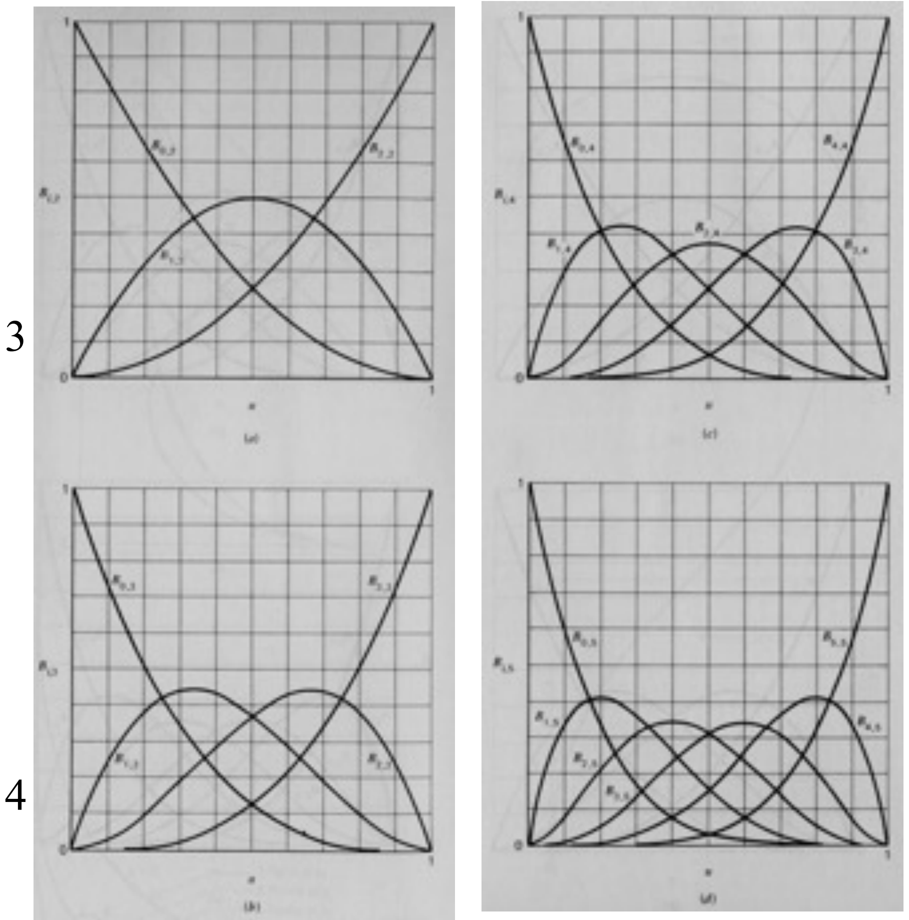


Bernstein Polynomials

- The blending functions are a special case of the Bernstein polynomials

$$b_{\mathrm{kd}}(u) = \frac{d!}{k!(d-k)!} u^k (1-u)^{d-k}$$

- These polynomials give the blending polynomials for any degree Bezier form
  - All roots at 0 and 1
  - For any degree they all sum to 1
  - They are all between 0 and 1 inside (0,1)



n = 3

n = 5

n = 4

n = 6

Bezier Curve Properties

- curve lies in the convex hull of the data
- variation diminishing
- symmetry
- affine invariant
- efficient evaluation and subdivision

Shirley Section 15.6.1 (p. 368)

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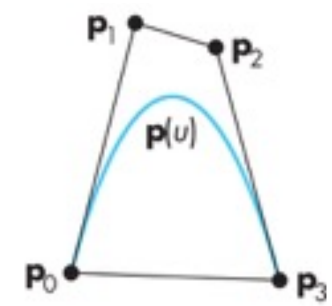

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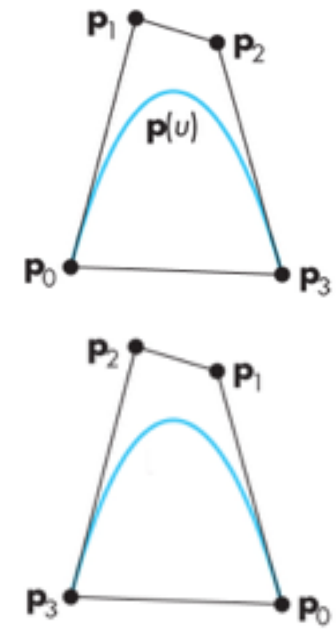

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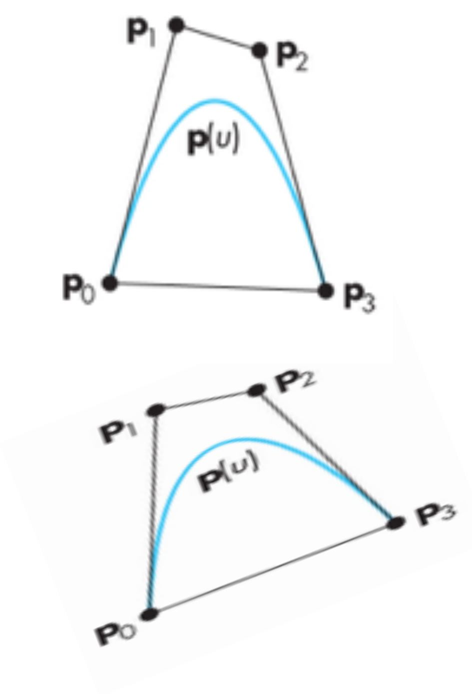

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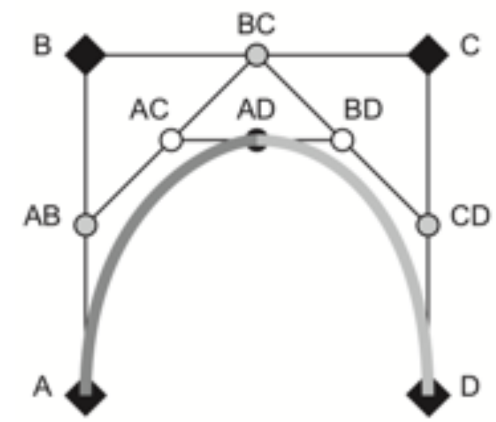

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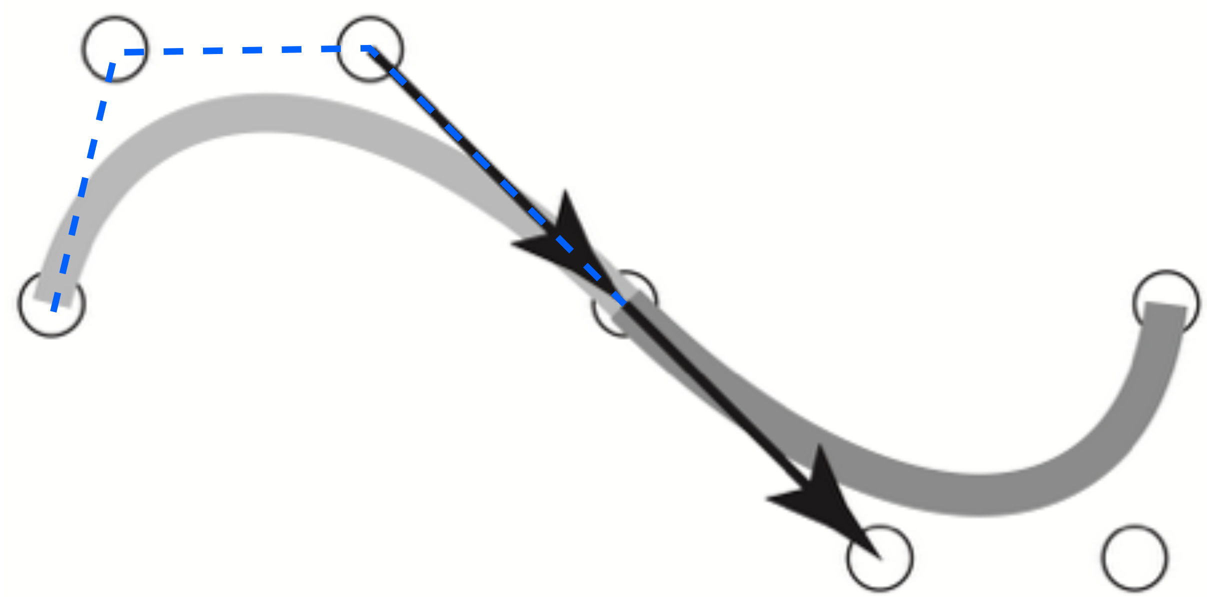
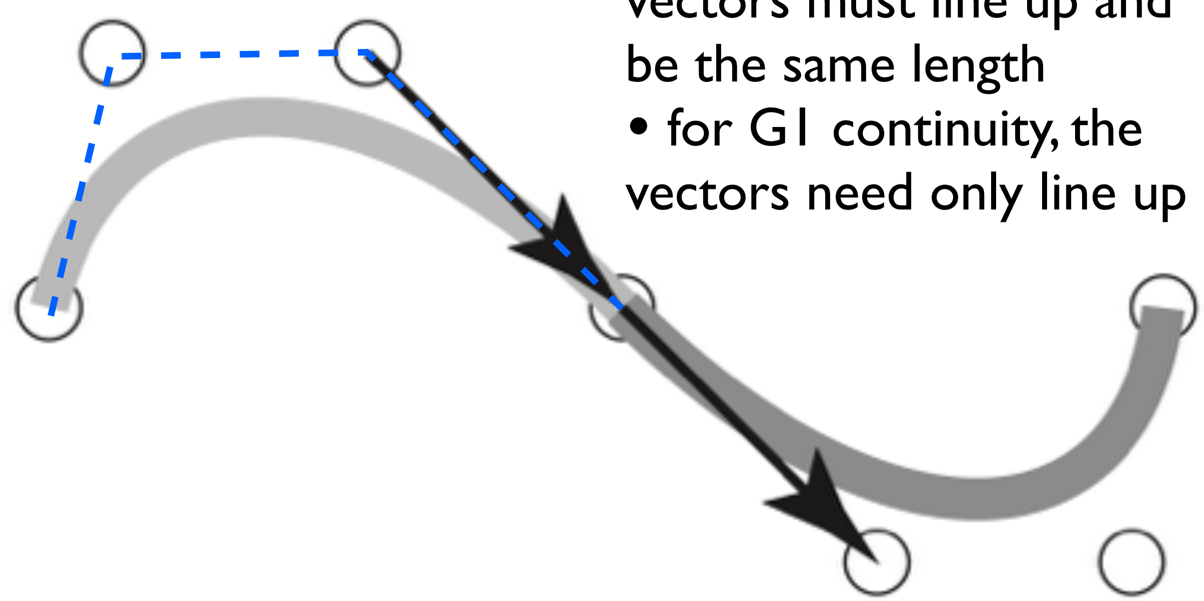
Joining Cubic Bezier Curves



for C1 continuity, the vectors must line up and be the same length
for G1 continuity, the vectors need only line up

Joining Cubic Bezier Curves



- for C1 continuity, the vectors must line up and be the same length
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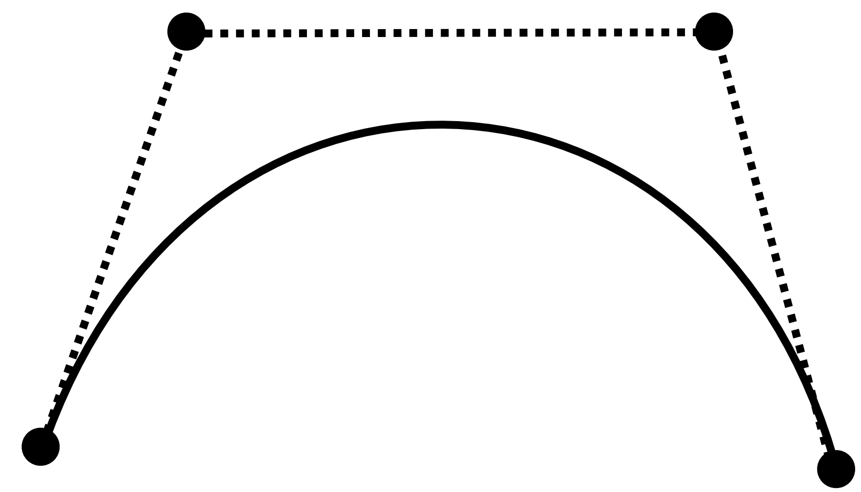
Evaluating p(u) geometrically



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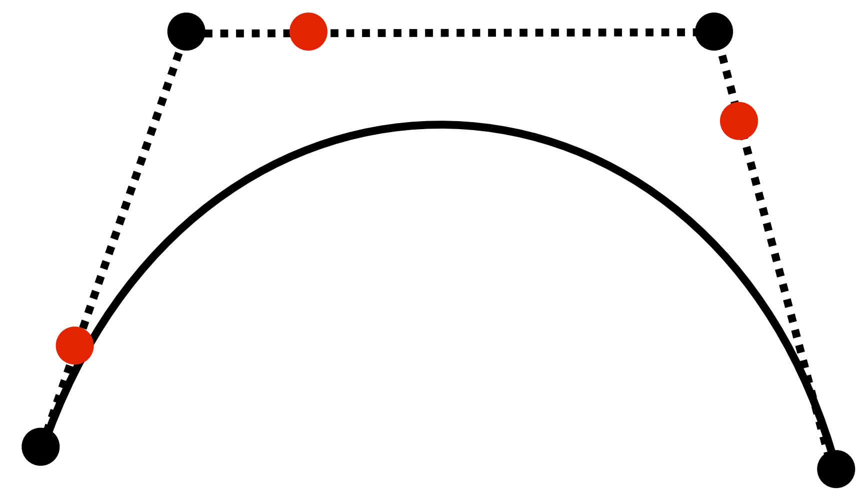

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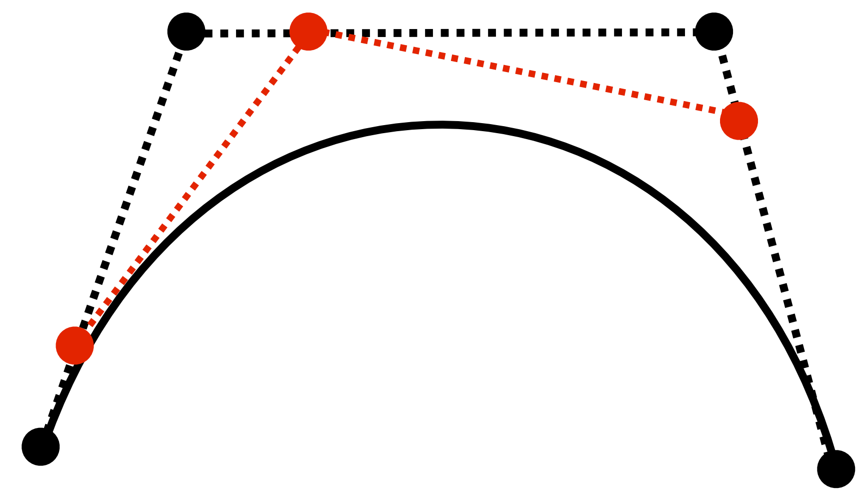

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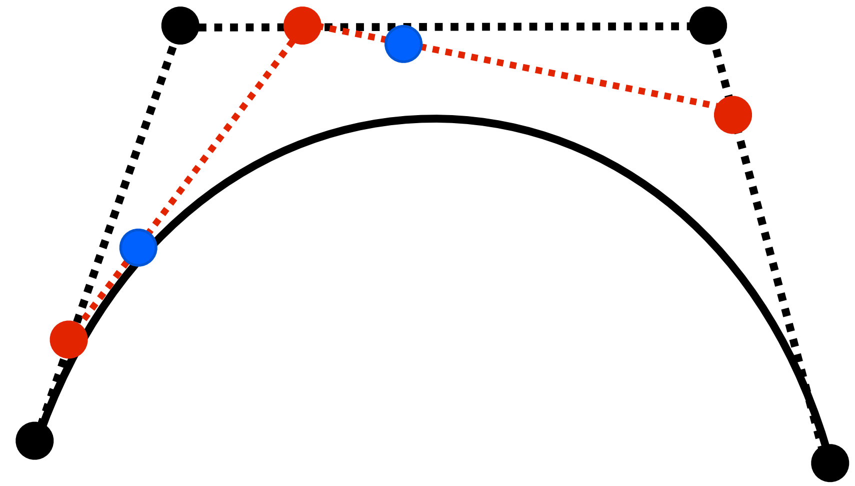

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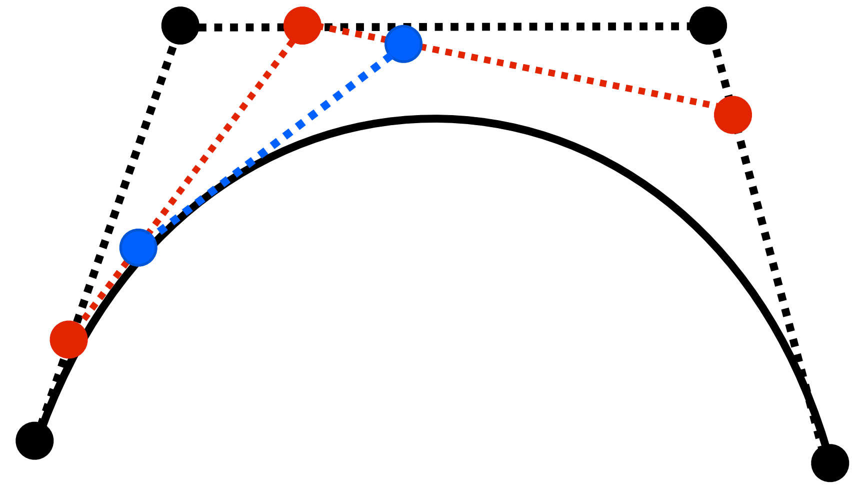

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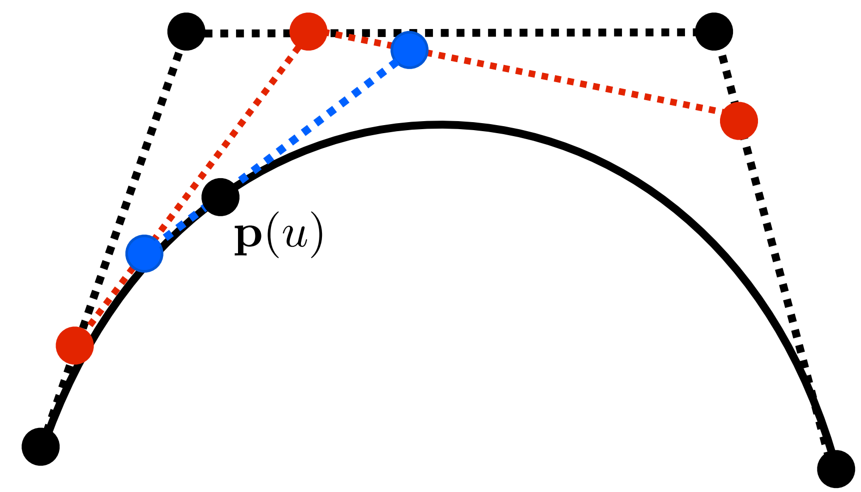

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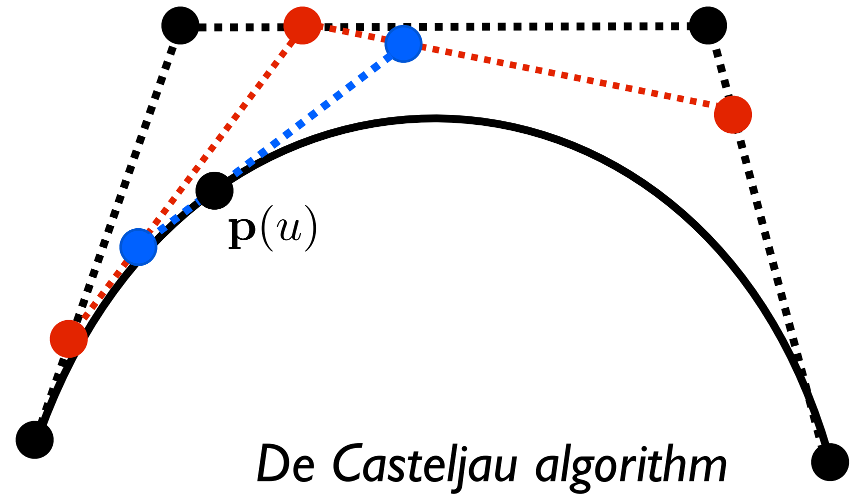
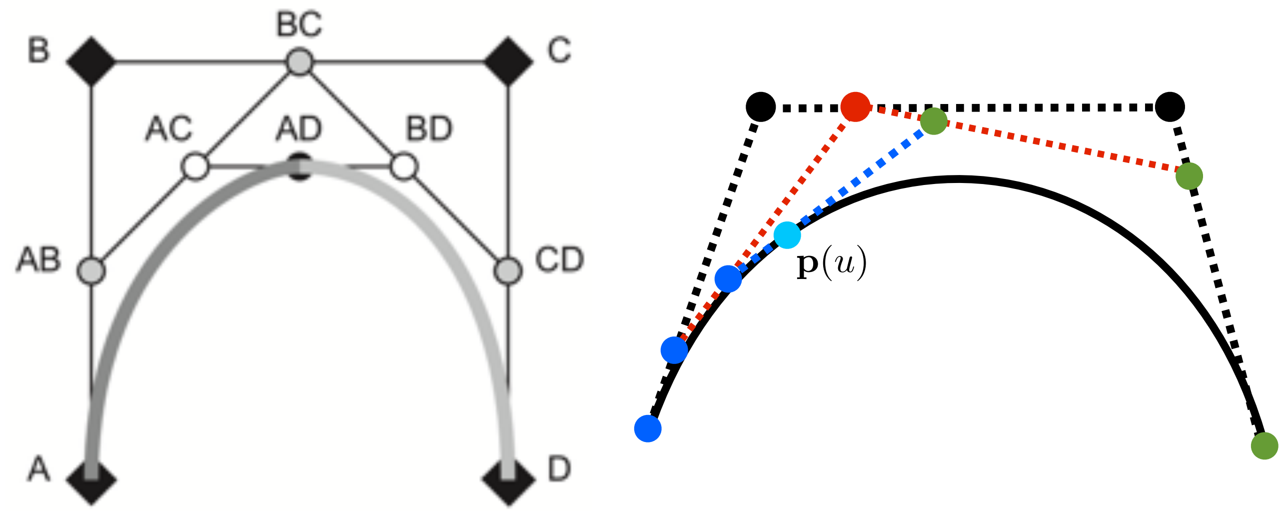


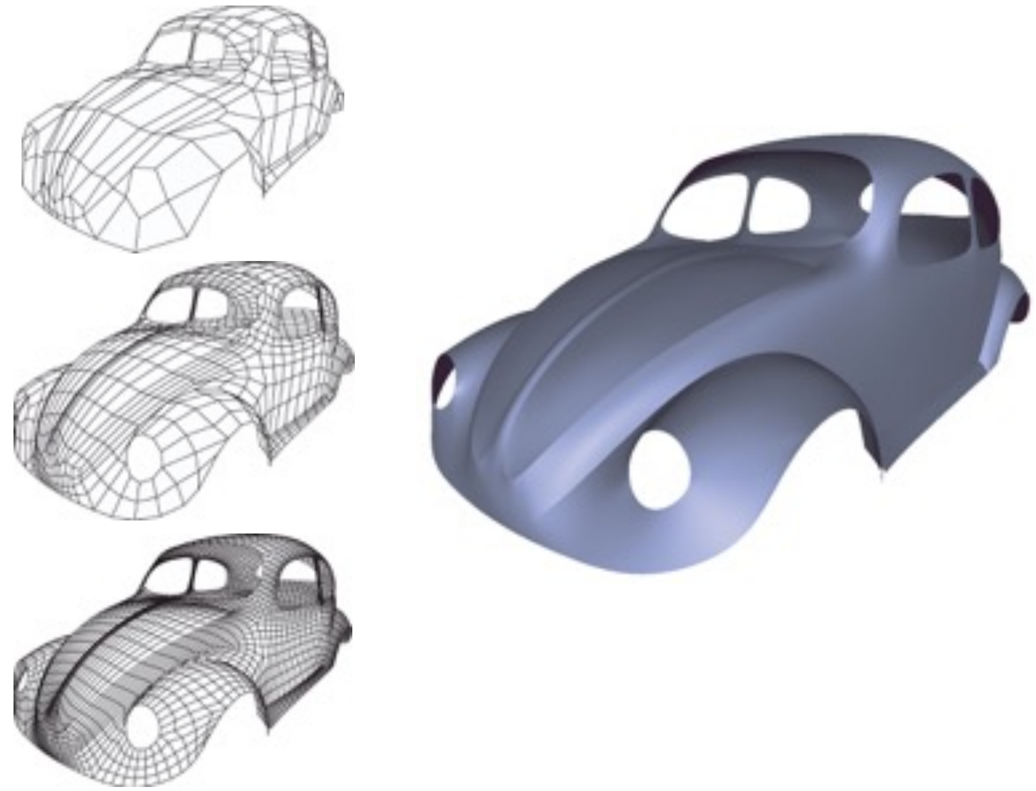
Kas-tell-joh

Bezier subdivision



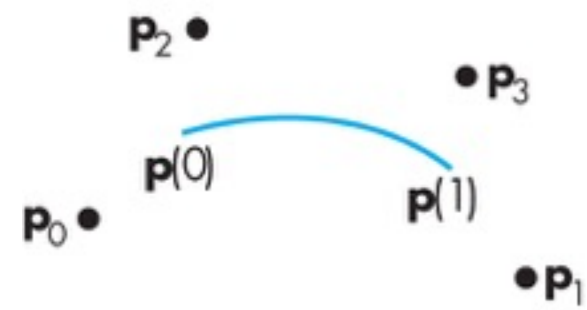
de Casteljau algorithm
Left: Subdivide the curve at the point u=.5
Right: Subdivide the curve at some other point u

Recursive Subdivision for Rendering



Cubic B-Splines

Cubic B-Splines



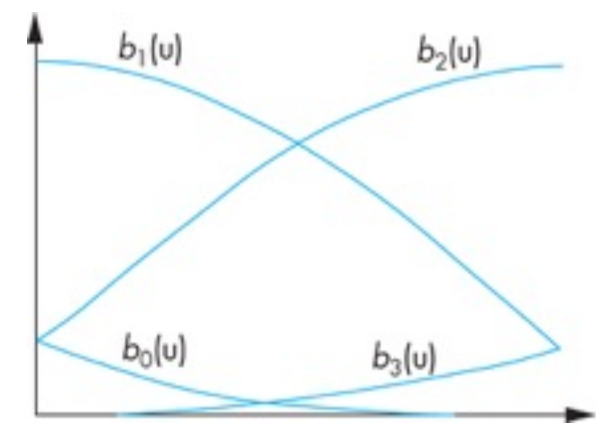
Spline blending functions

$$b_0(u) = \frac{1}{6}(1-u)^3$$

$$b_1(u) = \frac{1}{6}(4 - 6u^2 + 3u^3)$$

$$b_2(u) = \frac{1}{6}(1 + 3u + 3u^2 - 3u^3)$$

$$b_3(u) = \frac{1}{6}u^3$$

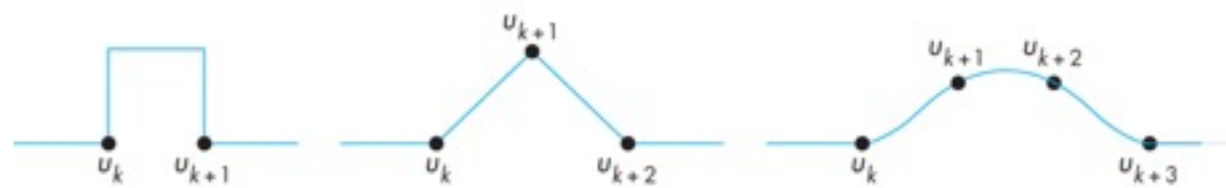


General Splines

- Defined recursively by *Cox-de Boor recursion formula*

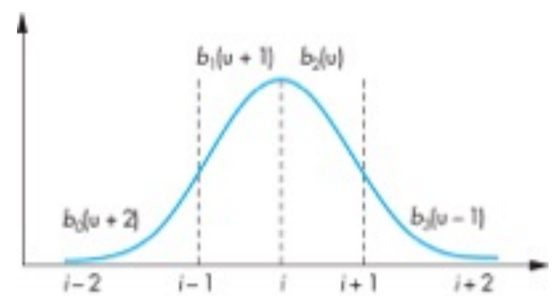
$$b_{j,0}(t) = \begin{cases} 1 & \text{if} \quad t_j \leq t \\ 0 & \text{otherwise} \end{cases}$$

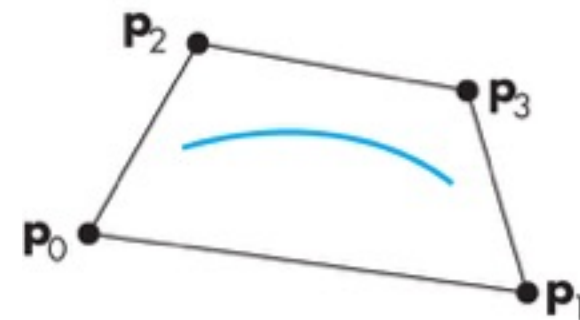
$$b_{j,n}(t) := \frac{t - t_j}{t_{j+n} - t_j} b_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} b_{j+1,n-1}(t)$$



Spline properties

Basis functions



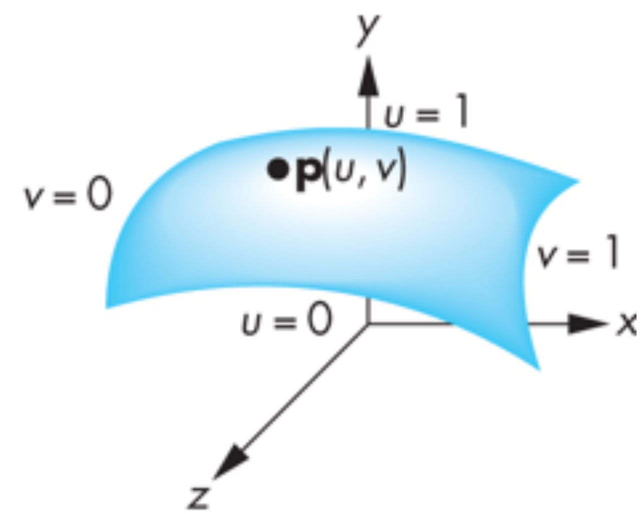


convexity

Surfaces

Parametric Surface

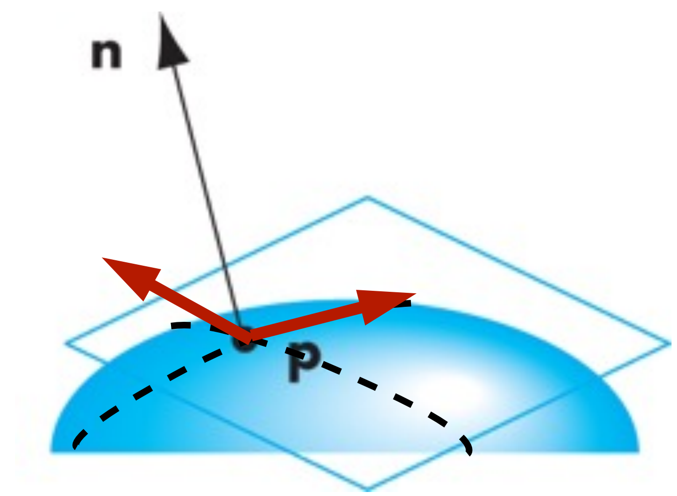
$$x = x(u, v)$$
$$y = y(u, v)$$
$$z = z(u, v)$$



Parametric Surface - tangent plane

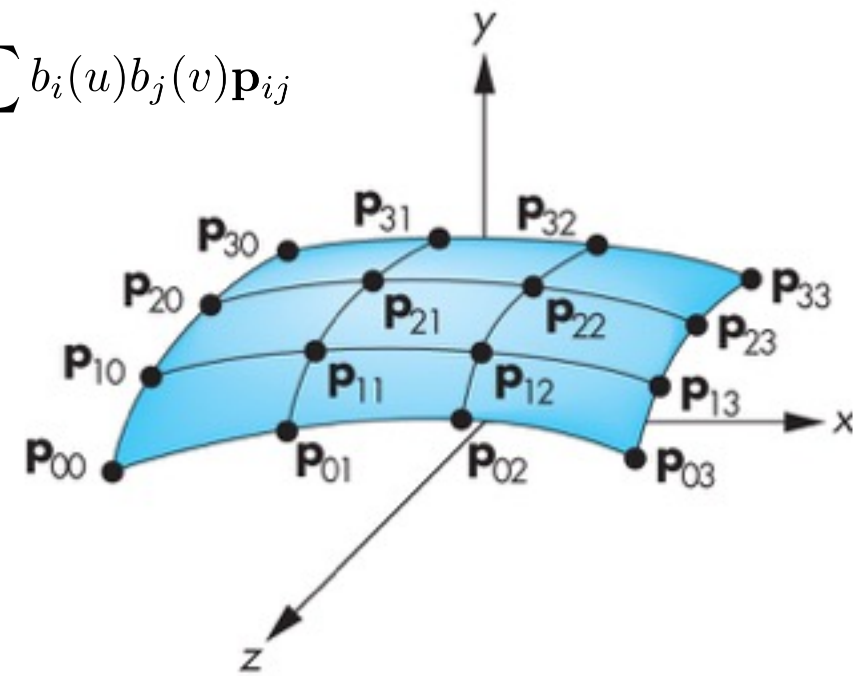
$$\mathbf{t}_u = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}$$

$$\mathbf{t}_v = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}$$



Bicubic Surface Patch

$$\mathbf{f}(u,v) = \sum_i \sum_j b_i(u) b_j(v) \mathbf{p}_{ij}$$



Bezier Surface Patch

$$\mathbf{f}(u,v) = \sum_i \sum_j b_i(u) b_j(v) \mathbf{p}_{ij}$$

Patch lies in
convex hull