Distributed Fractional Packing and Maximum Weighted b-Matching via Tail-Recursive Duality

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Fractional Packing

$$\max \sum_{j=1}^{m} w_j x_j$$

$$\text{subject to: } \sum_{j=1}^{m} A_{ij} x_j \le b_i \qquad (\forall i = 1 \ldots n)$$

$$x \in \mathbb{R}_+^m$$

Let δ be the maximum number of constraints in which a variable appears.

Maximum Weighted b-Matching

Hypergraph with edge weights *w*, vertex capacities *b*.

$$\max \sum_{e \in E} w_e x_e$$

$$\text{subject to: } \sum_{e \in E(u)} x_e \le b_u \quad (\forall u \in V)$$

$$x \in \mathbb{Z}_+^{|E|}$$

Let δ be the maximum number of constraints in which a variable appears, so δ is the hyperedge degree ($\delta = 2$ for graphs).

Distributed Computation

- The problem instance is represented by a graph where edges are packing variables and nodes are packing constraints.
- Computation takes place in rounds.
- In each round:
  - Each node exchanges O(1) messages with immediate neighbors,
  - then does some computation.
- **goal:** Finish in a poly-log number of rounds.

Related Work

Problem	Approx. ratio	Running time	Where
Max Weighted Matching on Graphs	$2+\varepsilon$	$O(\log(\varepsilon^{-1}) \log n)$	Lotker et al. 2008
	$1+\varepsilon$	$O(\varepsilon^{-4} \log^2 n)$	Lotker et al. 2008
	$1+\varepsilon$	$O(\varepsilon^{-2} + \varepsilon^{-1} \log(\varepsilon^{-1} n) \log n)$	Nieberg 2008
	2	$O(\log^2 n)$	($\varepsilon = 1$)
	2	$O(\log n)$	This work

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	$1+\varepsilon$	$O(\varepsilon^{-2} + \varepsilon^{-1} \log(\varepsilon^{-1} n) \log n)$	↑ Nieberg 2008
	2	$O(\log^2 n)$	↑ ($\varepsilon = 1$)
	2	$O(\log n)$	This work
Max Weighted Matching on Hypergraphs	$O(\delta)$ ($>\delta$)	$O(\log m)$	Kuhn et al. 2006
	δ	$O(\log^2 m)$	This work

Related Work

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	$1+\varepsilon$	$O(\varepsilon^{-2} + \varepsilon^{-1} \log(\varepsilon^{-1} n) \log n)$	↑ Nieberg 2008
	2	$O(\log^2 n)$	↑ ↑ ($\varepsilon = 1$)
	2	$O(\log n)$	This work
Max Weighted Matching on Hypergraphs	$O(\delta)$ ($>\delta$)	$O(\log m)$	Kuhn et al. 2006
	δ	$O(\log^2 m)$	This work
Fractional Packing $\delta = 2$	$O(1)$ (>2)	$O(\log m)$	Kuhn et al. 2006
	2	$O(\log m)$	This work

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Max Weighted Matching on Graphs	$2+\varepsilon$	$O(\log(\varepsilon^{-1}) \log n)$	Lotker et al. 2008
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	$1+\varepsilon$	$O(\varepsilon^{-2} + \varepsilon^{-1} \log(\varepsilon^{-1} n) \log n)$	↑ Nieberg 2008
	2	$O(\log^2 n)$	↑ ↑ (ε =1)
	2	$O(\log n)$	This work
Max Weighted Matching on Hypergraphs	$O(\delta)$ (>δ)	$O(\log m)$	Kuhn et al. 2006
	δ	$O(\log^2 m)$	This work
Fractional Packing $\delta = 2$	O(1) (>2)	$O(\log m)$	Kuhn et al. 2006
	2	$O(\log m)$	This work
Fractional Packing general δ	O(1) >12	$O(\log m)$	Kuhn et al. 2006
	δ	$O(\log^2 m)$	This work

Primal-Dual
δ-approximation algorithm

Packing

$$\max \sum_{j=1}^{m} w_j x_j$$

subject to:

$$\sum_{j=1}^{m} A_{ij} x_j \le b_i \quad (\forall i = 1 \ldots n)$$

$$x \in \mathbb{R}_+^m$$

Covering

$$\min \sum_{i=1}^{n} b_i y_i$$

subject to:

$$\sum_{i=1}^{n} A_{ij} y_i \ge w_j \quad (\forall j = 1 \ldots m)$$

$$y \in \mathbb{R}_+^n$$

- Compute a solution for the dual Covering Problem.
- Use the dual solution to compute a solution for the primal Packing Problem.

δ-approximation for Covering

1. Let $y \leftarrow 0$.

2. While $\exists\, j$ such that $\sum_{i=1}^{n} A_{ij} y_i < w_j$ do:

3. Let $\beta = \underbrace{\left(w_j - \sum_{i=1}^{n} A_{ij} y_i \right)}_{\textit{slack}} \cdot \underbrace{\min_i \left(b_i / A_{ij} \right)}_{\text{cost to reduce slack by 1 using the cheapest variable}}$

OPT cost to satisfy the constraint given the current solution y

4. Raise each y_i (with $A_{ij} \neq 0$) by β / b_i.

5. Return y.

$$\min \sum_{i=1}^{n} b_i y_i$$

subject to:

$$\sum_{i=1}^{n} A_{ij} y_i \geq w_j \qquad \left(\forall j = 1 \ldots m \right)$$

$$y \in \mathbb{R}_+^n$$

Primal-Dual Attempt

1. Let $x, y \leftarrow 0$.
2. While $\exists\, j$ such that $\sum_{i=1}^{n} A_{ij} y_i < w_j$ do:
3. Do a "step" to satisfy this covering constraint.
4. Raise x_j maximally without violating any packing constraint.
5. Return x, y.

$$\max \sum_{j=1}^{m} w_j x_j$$

subject to:

$$\sum_{j=1}^{m} A_{ij} x_j \le b_i \quad (\forall i = 1 \ldots n)$$

$$x \in \mathbb{R}_+^m$$

$$\min \sum_{i=1}^{n} b_i y_i$$

subject to:

$$\sum_{i=1}^{n} A_{ij} y_i \ge w_j \quad (\forall j = 1 \ldots m)$$

$$y \in \mathbb{R}_+^n$$

Primal-Dual Attempt

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3. Do a "step" to satisfy this covering constraint.
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5. Return x, y.

max $\sum_{j=1}^{m} w_j x_j$

subject to:

$$\sum_{j=1}^{m} A_{ij} x_j \le b_i \quad (\forall i = 1 \ldots n)$$

$$x \in \mathbb{R}_+^m$$

min $\sum_{i=1}^{n} b_i y_i$

subject to:

$$\sum_{i=1}^{n} A_{ij} y_i \ge w_j \quad (\forall j = 1 \ldots m)$$

$$y \in \mathbb{R}_+^n$$

Note δ = 2

$$\begin{aligned} \min\ & y_1 + y_2 + y_3 \\ \text{s.t.}\ & y_1 + y_2 \ge 1 \\ & y_1 + y_3 \ge 5 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

$$\begin{aligned} \max\ & x_{12} + 5x_{13} \\ \text{s.t.}\ & x_{12} + x_{13} \le 1 \\ & x_{12} \le 1 \\ & x_{13} \le 1 \\ & x_{12}, x_{13} \ge 0 \end{aligned}$$

Note $\delta = 2$

$$\begin{aligned} \min\ & y_1 + y_2 + y_3 \\ \text{s.t.}\ & y_1 + y_2 \geq 1 \\ & y_1 + y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

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choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

↓

↓

raise x_{12} maximally

$x_{12} = 1$

Note $\delta = 2$

$$\min\ y_1 + y_2 + y_3$$
$$\text{s.t. } y_1 + y_2 \ge 1$$
$$y_1 + y_3 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

$$\max\ x_{12} + 5x_{13}$$
$$\text{s.t. } x_{12} + x_{13} \le 1$$
$$x_{12} \le 1$$
$$x_{13} \le 1$$
$$x_{12}, x_{13} \ge 0$$

choose $y_1 + y_2 \ge 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

raise x_{12} maximally

$x_{12} = 1$

$$\min\ y_1 + y_2 + y_3$$
$$\text{s.t. } y_1 + y_2 \ge -1$$
$$y_1 + y_3 \ge 4$$

$$\max\ -x_{12} + 4x_{13}$$
$$\text{s.t. } x_{12} + x_{13} \le 1$$

Note $\delta = 2$

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \geq 1,\ y_1 + y_3 \geq 5,\ y_1, y_2, y_3 \geq 0$$

$$\max\ x_{12} + 5x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \leq 1,\ x_{12} \leq 1,\ x_{13} \leq 1,\ x_{12}, x_{13} \geq 0$$

choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

raise x_{12} maximally

$x_{12} = 1$

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \geq -1,\ y_1 + y_3 \geq 4$$

$$\max\ -x_{12} + 4x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \leq 1$$

choose $y_1 + y_3 \geq 4$

step $y_1 += 4,\ y_3 += 4$

primal cost += 8

raise x_{13} maximally

(it remains 0)

Note $\delta = 2$

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \ge 1,\ y_1 + y_3 \ge 5,\ y_1, y_2, y_3 \ge 0$$

$$\max\ x_{12} + 5x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \le 1,\ x_{12} \le 1,\ x_{13} \le 1,\ x_{12}, x_{13} \ge 0$$

choose $y_1 + y_2 \ge 1$
step $y_1 += 1,\ y_2 += 1$
primal cost += 2

raise x_{12} maximally
$x_{12} = 1$

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \ge -1,\ y_1 + y_3 \ge 4$$

$$\max\ -x_{12} + 4x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \le 1$$

choose $y_1 + y_3 \ge 4$
step $y_1 += 4,\ y_3 += 4$
primal cost += 8

raise x_{13} maximally
(it remains 0)

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \ge -5,\ y_1 + y_3 \ge -4$$

$$\max\ -5x_{12} - 4x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \le 1$$

Note $\delta = 2$

$\min\ y_1 + y_2 + y_3$	$\max\ x_{12} + 5x_{13}$
s.t. $y_1 + y_2 \geq 1$	s.t. $x_{12} + x_{13} \leq 1$
$y_1 + y_3 \geq 5$	$x_{12} \leq 1$
$y_1, y_2, y_3 \geq 0$	$x_{13} \leq 1$
	$x_{12}, x_{13} \geq 0$

choose $y_1 + y_2 \geq 1$
step $y_1 {+}{=} 1,\ y_2 {+}{=} 1$
primal cost += 2

raise x_{12} maximally
$x_{12} = 1$

$\min\ y_1 + y_2 + y_3$	$\max\ -x_{12} + 4x_{13}$
s.t. $y_1 + y_2 \geq -1$	s.t. $x_{12} + x_{13} \leq 1$
$y_1 + y_3 \geq 4$	

choose $y_1 + y_3 \geq 4$
step $y_1 {+}{=} 4,\ y_3 {+}{=} 4$
primal cost += 8

raise x_{13} maximally
(it remains 0)

$\min\ y_1 + y_2 + y_3$	$\max\ -5x_{12} - 4x_{13}$
s.t. $y_1 + y_2 \geq -5$	s.t. $x_{12} + x_{13} \leq 1$
$y_1 + y_3 \geq -4$	

Final primal solution:
$y_1 = 5, y_2 = 1, y_3 = 4$
primal cost = 10

Final dual solution:
$x_{12} = 1, x_{13} = 0$
dual cost = 1

Note $\delta = 2$

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \ge 1,\quad y_1 + y_3 \ge 5,\quad y_1, y_2, y_3 \ge 0$$

$$\max\ x_{12} + 5x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \le 1,\quad x_{12} \le 1,\quad x_{13} \le 1,\quad x_{12}, x_{13} \ge 0$$

choose $y_1 + y_2 \ge 1$

step $y_1 {+}{=} 1,\ y_2 {+}{=} 1$

primal cost += 2

raise x_{12} maximally

$x_{12} = 1$

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \ge -1,\quad y_1 + y_3 \ge 4$$

$$\max\ -x_{12} + 4x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \le 1$$

choose $y_1 + y_3 \ge 4$

step $y_1 {+}{=} 4,\ y_3 {+}{=} 4$

primal cost += 8

raise x_{13} maximally

(it remains 0)

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \ge -5,\quad y_1 + y_3 \ge -4$$

$$\max\ -5x_{12} - 4x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \le 1$$

Not 2-approximate

Final primal solution:

$y_1 = 5, y_2 = 1, y_3 = 4$

primal cost = 10

Final dual solution:

$x_{12} = 1, x_{13} = 0$

dual cost = 1

Primal-Dual Algorithm

1. Let $x, y \leftarrow 0$.
2. While $\exists\, j$ such that $\sum_{i=1}^{n} A_{ij} y_i < w_j$ do:
3. Do a "step" to satisfy this covering constraint..
4. For each j for which a step was done (line 2) in reverse order do:
5. Raise x_j maximally without violating any packing constraint.
6. Return x, y.

max $\sum_{j=1}^{m} w_j x_j$

subject to:

$$\sum_{j=1}^{m} A_{ij} x_j \le b_i \qquad (\forall i = 1 \ldots n)$$

$$x \in \mathbb{R}_{+}^{m}$$

min $\sum_{i=1}^{n} b_i y_i$

subject to:

$$\sum_{i=1}^{n} A_{ij} y_i \ge w_j \qquad (\forall j = 1 \ldots m)$$

$$y \in \mathbb{R}_{+}^{n}$$

Note δ = 2

$$
\begin{aligned}
\min\ & y_1 + y_2 + y_3 \\
\text{s.t. } & y_1 + y_2 \geq 1 \\
& y_1 + y_3 \geq 5 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Note $\delta = 2$

Execution starts here. Follow the arrows

$$\begin{aligned} \min\ & y_1 + y_2 + y_3 \\ \text{s.t.}\ & y_1 + y_2 \geq 1 \\ & y_1 + y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ \ y_2 += 1$

primal cost += 2

Note $\delta = 2$

Execution starts here.
Follow the arrows

$$\begin{aligned}\min\ & y_1 + y_2 + y_3 \\ \text{s.t.}\ & y_1 + y_2 \ge 1 \\ & y_1 + y_3 \ge 5 \\ & y_1, y_2, y_3 \ge 0\end{aligned}$$

choose $y_1 + y_2 \ge 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

$$\begin{aligned}\min\ & y'_1 + y'_2 + y'_3 \\ \text{s.t.}\ & y'_1 + y'_2 \ge -1 \\ & y'_1 + y'_3 \ge 4 \\ & y'_1, y'_2, y'_3 \ge 0\end{aligned}$$

Note $\delta = 2$

Execution starts here.
Follow the arrows

$$\begin{aligned} \min\ & y_1 + y_2 + y_3 \\ \text{s.t.}\ & y_1 + y_2 \geq 1 \\ & y_1 + y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

$$\begin{aligned} \min\ & y'_1 + y'_2 + y'_3 \\ \text{s.t.}\ & y'_1 + y'_2 \geq -1 \\ & y'_1 + y'_3 \geq 4 \\ & y'_1, y'_2, y'_3 \geq 0 \end{aligned}$$

choose $y'_1 + y'_3 \geq 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

Note $\delta = 2$

Execution starts here.
Follow the arrows

$$\begin{aligned} \min\ & y_1 + y_2 + y_3 \\ \text{s.t.}\ & y_1 + y_2 \geq 1 \\ & y_1 + y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

$$\begin{aligned} \min\ & y'_1 + y'_2 + y'_3 \\ \text{s.t.}\ & y'_1 + y'_2 \geq -1 \\ & y'_1 + y'_3 \geq 4 \\ & y'_1, y'_2, y'_3 \geq 0 \end{aligned}$$

choose $y'_1 + y'_3 \geq 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

$$\begin{aligned} \min\ & y''_1 + y''_2 + y''_3 \\ \text{s.t.}\ & y''_1 + y''_2 \geq -5 \\ & y''_1 + y''_3 \geq -4 \\ & y''_1, y''_2, y''_3 \geq 0 \end{aligned}$$

Note $\delta = 2$

Execution starts here.
Follow the arrows

$$\min\ y_1 + y_2 + y_3$$
$$\text{s.t. } y_1 + y_2 \geq 1$$
$$y_1 + y_3 \geq 5$$
$$y_1, y_2, y_3 \geq 0$$

choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

$$\min\ y'_1 + y'_2 + y'_3$$
$$\text{s.t. } y'_1 + y'_2 \geq -1$$
$$y'_1 + y'_3 \geq 4$$
$$y'_1, y'_2, y'_3 \geq 0$$

choose $y'_1 + y'_3 \geq 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

$$\min\ y''_1 + y''_2 + y''_3$$
$$\text{s.t. } y''_1 + y''_2 \geq -5$$
$$y''_1 + y''_3 \geq -4$$
$$y''_1, y''_2, y''_3 \geq 0$$

Base case

$$y''_1 = y''_2 = y''_3 = 0$$

Final primal solution:

$$y_1 = 5, y_2 = 1, y_3 = 4$$

primal cost = 10

Note $\delta = 2$

Execution starts here. Follow the arrows

$$\min\ y_1 + y_2 + y_3$$
$$\text{s.t. } y_1 + y_2 \ge 1$$
$$y_1 + y_3 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

choose $y_1 + y_2 \ge 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

$$\min\ y'_1 + y'_2 + y'_3$$
$$\text{s.t. } y'_1 + y'_2 \ge -1$$
$$y'_1 + y'_3 \ge 4$$
$$y'_1, y'_2, y'_3 \ge 0$$

choose $y'_1 + y'_3 \ge 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

$$\min\ y''_1 + y''_2 + y''_3$$
$$\text{s.t. } y''_1 + y''_2 \ge -5$$
$$y''_1 + y''_3 \ge -4$$
$$y''_1, y''_2, y''_3 \ge 0$$

$$\max\ -5x''_{12} - 4x''_{13}$$
$$\text{s.t. } x''_{12} + x''_{13} \le 1$$
$$x''_{12}, x''_{13} \ge 0$$

Base case

$y''_1 = y''_2 = y''_3 = 0$

$x''_{12} = x''_{13} = 0$

dual cost = 0

Final primal solution:

$y_1 = 5, y_2 = 1, y_3 = 4$

primal cost = 10

Note $\delta = 2$

Execution starts here.
Follow the arrows

$$\min\ y_1 + y_2 + y_3$$
$$\text{s.t. } y_1 + y_2 \geq 1$$
$$y_1 + y_3 \geq 5$$
$$y_1, y_2, y_3 \geq 0$$

choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

$$\min\ y'_1 + y'_2 + y'_3$$
$$\text{s.t. } y'_1 + y'_2 \geq -1$$
$$y'_1 + y'_3 \geq 4$$
$$y'_1, y'_2, y'_3 \geq 0$$

choose $y'_1 + y'_3 \geq 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

raise x'_{13} maximally (it becomes 1)

$$\min\ y''_1 + y''_2 + y''_3$$
$$\text{s.t. } y''_1 + y''_2 \geq -5$$
$$y''_1 + y''_3 \geq -4$$
$$y''_1, y''_2, y''_3 \geq 0$$

$$\max\ -5x''_{12} - 4x''_{13}$$
$$\text{s.t. } x''_{12} + x''_{13} \leq 1$$
$$x''_{12}, x''_{13} \geq 0$$

Base case

$y''_1 = y''_2 = y''_3 = 0$

$x''_{12} = x''_{13} = 0$

dual cost = 0

Final primal solution:

$y_1 = 5, y_2 = 1, y_3 = 4$

primal cost = 10

Note $\delta = 2$

Execution starts here. Follow the arrows

$$\begin{aligned} \min\ & y_1 + y_2 + y_3 \\ \text{s.t.}\ & y_1 + y_2 \ge 1 \\ & y_1 + y_3 \ge 5 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

choose $y_1 + y_2 \ge 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

$$\begin{aligned} \min\ & y'_1 + y'_2 + y'_3 \\ \text{s.t.}\ & y'_1 + y'_2 \ge -1 \\ & y'_1 + y'_3 \ge 4 \\ & y'_1, y'_2, y'_3 \ge 0 \end{aligned}$$

$$\begin{aligned} \max\ & -x'_{12} + 4x'_{13} \\ \text{s.t.}\ & x'_{12} + x'_{13} \le 1 \\ & x'_{12}, x'_{13} \ge 0 \end{aligned}$$

$x'_{12} = 0,\ x'_{13} = 1$

dual cost += 4

choose $y'_1 + y'_3 \ge 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

raise x'_{13} maximally (it becomes 1)

$$\begin{aligned} \min\ & y''_1 + y''_2 + y''_3 \\ \text{s.t.}\ & y''_1 + y''_2 \ge -5 \\ & y''_1 + y''_3 \ge -4 \\ & y''_1, y''_2, y''_3 \ge 0 \end{aligned}$$

$$\begin{aligned} \max\ & -5x''_{12} - 4x''_{13} \\ \text{s.t.}\ & x''_{12} + x''_{13} \le 1 \\ & x''_{12}, x''_{13} \ge 0 \end{aligned}$$

Base case

$y''_1 = y''_2 = y''_3 = 0$

$x''_{12} = x''_{13} = 0$

dual cost = 0

Final primal solution:

$y_1 = 5, y_2 = 1, y_3 = 4$

primal cost = 10

Note $\delta = 2$

Execution starts here.
Follow the arrows

$$\begin{aligned} \min\ & y_1 + y_2 + y_3 \\ \text{s.t. } & y_1 + y_2 \geq 1 \\ & y_1 + y_3 \geq 5 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

raise x_{12} maximally

(it remains 0)

$$\begin{aligned} \min\ & y'_1 + y'_2 + y'_3 \\ \text{s.t. } & y'_1 + y'_2 \geq -1 \\ & y'_1 + y'_3 \geq 4 \\ & y'_1, y'_2, y'_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & -x'_{12} + 4x'_{13} \\ \text{s.t. } & x'_{12} + x'_{13} \leq 1 \\ & x'_{12}, x'_{13} \geq 0 \end{aligned}$$

$x'_{12} = 0,\ x'_{13} = 1$

dual cost += 4

choose $y'_1 + y'_3 \geq 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

raise x'_{13} maximally

(it becomes 1)

$$\begin{aligned} \min\ & y''_1 + y''_2 + y''_3 \\ \text{s.t. } & y''_1 + y''_2 \geq -5 \\ & y''_1 + y''_3 \geq -4 \\ & y''_1, y''_2, y''_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & -5x''_{12} - 4x''_{13} \\ \text{s.t. } & x''_{12} + x''_{13} \leq 1 \\ & x''_{12}, x''_{13} \geq 0 \end{aligned}$$

Base case

$y''_1 = y''_2 = y''_3 = 0$

$x''_{12} = x''_{13} = 0$

dual cost = 0

Final primal solution:

$y_1 = 5, y_2 = 1, y_3 = 4$

primal cost = 10

Note $\delta = 2$

Execution starts here. Follow the arrows

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t. } y_1 + y_2 \ge 1,\ y_1 + y_3 \ge 5,\ y_1, y_2, y_3 \ge 0$$

$$\max\ x_{12} + 5x_{13} \quad \text{s.t. } x_{12} + x_{13} \le 1,\ x_{12} \le 1,\ x_{13} \le 1,\ x_{12}, x_{13} \ge 0$$

$x_{12} = 0, x_{13} = 1$

dual cost += 1

choose $y_1 + y_2 \ge 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

raise x_{12} maximally (it remains 0)

dual cost increases by 1 because w_{13} increases by 1 and $x_{13}=1$

$$\min\ y'_1 + y'_2 + y'_3 \quad \text{s.t. } y'_1 + y'_2 \ge -1,\ y'_1 + y'_3 \ge 4,\ y'_1, y'_2, y'_3 \ge 0$$

$$\max\ -x'_{12} + 4x'_{13} \quad \text{s.t. } x'_{12} + x'_{13} \le 1,\ x'_{12}, x'_{13} \ge 0$$

$x'_{12} = 0,\ x'_{13} = 1$

dual cost += 4

choose $y'_1 + y'_3 \ge 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

raise x'_{13} maximally (it becomes 1)

$$\min\ y''_1 + y''_2 + y''_3 \quad \text{s.t. } y''_1 + y''_2 \ge -5,\ y''_1 + y''_3 \ge -4,\ y''_1, y''_2, y''_3 \ge 0$$

$$\max\ -5x''_{12} - 4x''_{13} \quad \text{s.t. } x''_{12} + x''_{13} \le 1,\ x''_{12}, x''_{13} \ge 0$$

Base case

$y''_1 = y''_2 = y''_3 = 0$

$x''_{12} = x''_{13} = 0$

dual cost = 0

Final primal solution:

$y_1 = 5, y_2 = 1, y_3 = 4$

primal cost = 10

Note $\delta = 2$

Execution starts here. Follow the arrows

Final dual solution:

$x_{12} = 0, x_{13} = 1$

dual cost = 5

$$\min\ y_1 + y_2 + y_3 \quad \text{s.t.}\ y_1 + y_2 \geq 1,\ y_1 + y_3 \geq 5,\ y_1, y_2, y_3 \geq 0$$

$$\max\ x_{12} + 5x_{13} \quad \text{s.t.}\ x_{12} + x_{13} \leq 1,\ x_{12} \leq 1,\ x_{13} \leq 1,\ x_{12}, x_{13} \geq 0$$

$x_{12} = 0, x_{13} = 1$

dual cost += 1

choose $y_1 + y_2 \geq 1$

step $y_1 += 1,\ y_2 += 1$

primal cost += 2

raise x_{12} maximally (it remains 0)

dual cost increases by 1 because w_{13} increases by 1 and $x_{13}=1$

$$\min\ y'_1 + y'_2 + y'_3 \quad \text{s.t.}\ y'_1 + y'_2 \geq -1,\ y'_1 + y'_3 \geq 4,\ y'_1, y'_2, y'_3 \geq 0$$

$$\max\ -x'_{12} + 4x'_{13} \quad \text{s.t.}\ x'_{12} + x'_{13} \leq 1,\ x'_{12}, x'_{13} \geq 0$$

$x'_{12} = 0,\ x'_{13} = 1$

dual cost += 4

choose $y'_1 + y'_3 \geq 4$

step $y'_1 += 4,\ y'_3 += 4$

primal cost += 8

raise x'_{13} maximally (it becomes 1)

$$\min\ y''_1 + y''_2 + y''_3 \quad \text{s.t.}\ y''_1 + y''_2 \geq -5,\ y''_1 + y''_3 \geq -4,\ y''_1, y''_2, y''_3 \geq 0$$

$$\max\ -5x''_{12} - 4x''_{13} \quad \text{s.t.}\ x''_{12} + x''_{13} \leq 1,\ x''_{12}, x''_{13} \geq 0$$

Base case

$y''_1 = y''_2 = y''_3 = 0$

$x''_{12} = x''_{13} = 0$

dual cost = 0

Final primal solution:

$y_1 = 5, y_2 = 1, y_3 = 4$

primal cost = 10

Recap of the algorithm

- Perform steps to satisfy covering constraints $\Longleftrightarrow$ record the *partial* order of the covering constraints for which the algorithm performed a step.
- Consider the corresponding packing variables in a *valid reverse order*, and raising them maximally.

Partial order

The order in which steps to satisfy covering constraints does not matter for covering constraints that do no share a variable.

Since the order is partial, there might be more than one valid reverse orders. Any one of them works!

Max Weighted b-matching

- Raise x_j maximally $\Longleftrightarrow$ $x_j = \min_i \left(b_i - \sum_{j=1}^{m} A_{ij} x_j \right)$
- If $A_{ij} = 0/1$ and $b_i \in \mathbb{Z}_+$ then $x_j \in \mathbb{Z}_+$

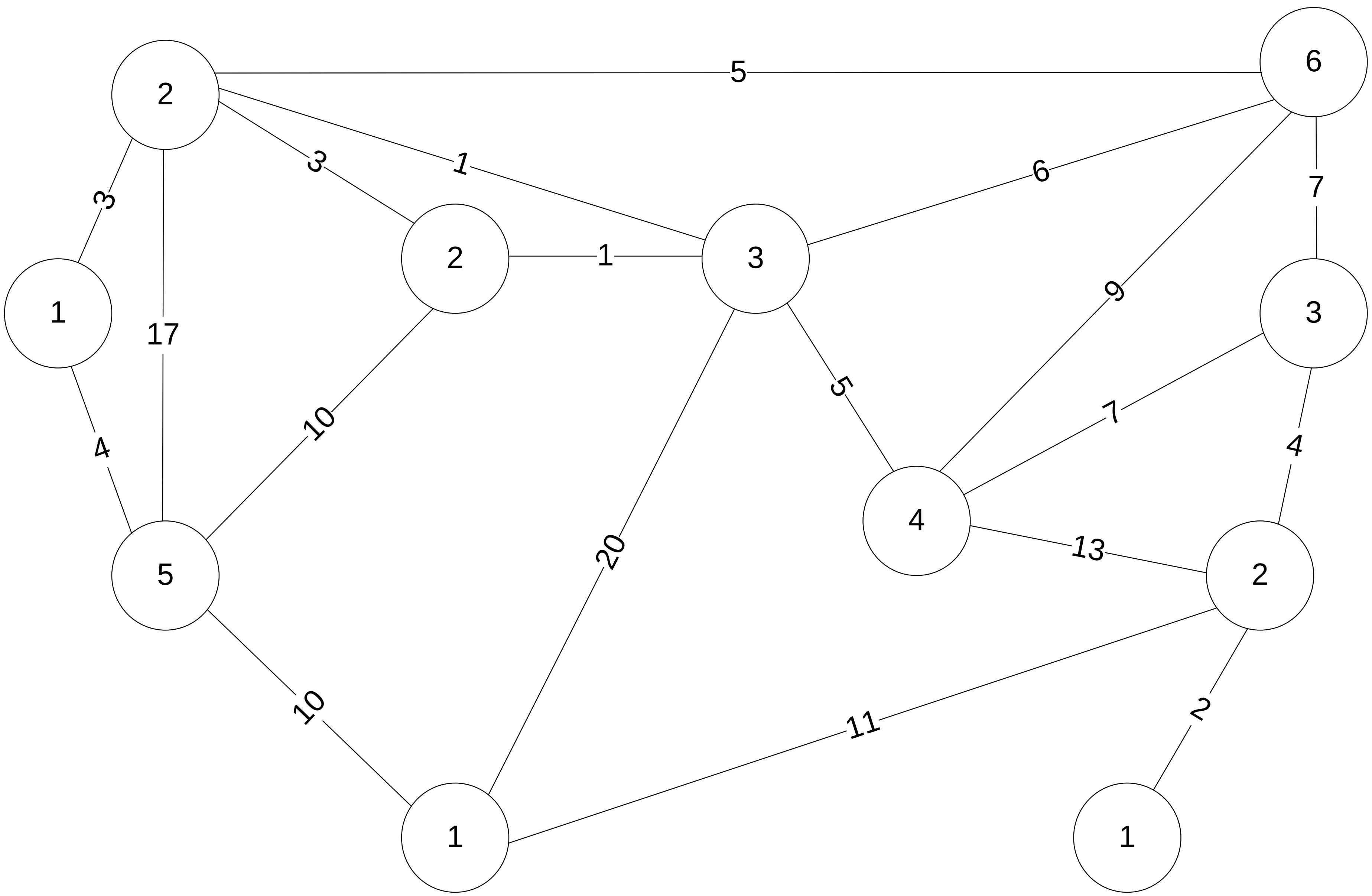
$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{subject to:} \quad & \sum_{e \in E(u)} x_e \le b_u \qquad (\forall u \in V) \\ & x \in \mathbb{Z}_+^{|E|} \end{aligned}$$

Distributed Computation

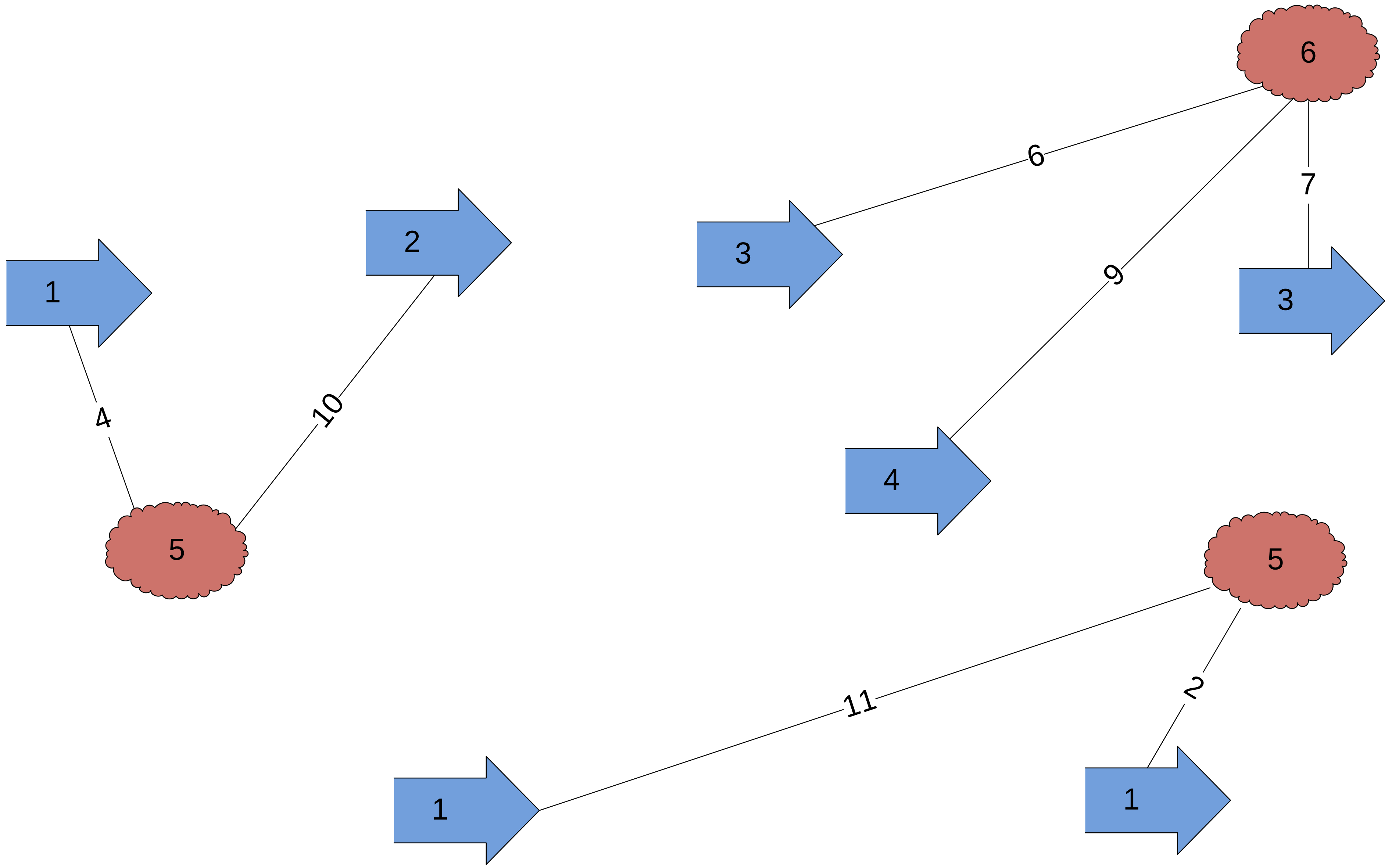
- Compute solution for the covering problem [KY 2009].
- Start computing solution for the packing (matching) problem as soon as possible.

(an edge can compute its packing variable as long as *all* of its adjacent edges are covered and they are not waiting for their adjacent edges)

- Distributed covering:

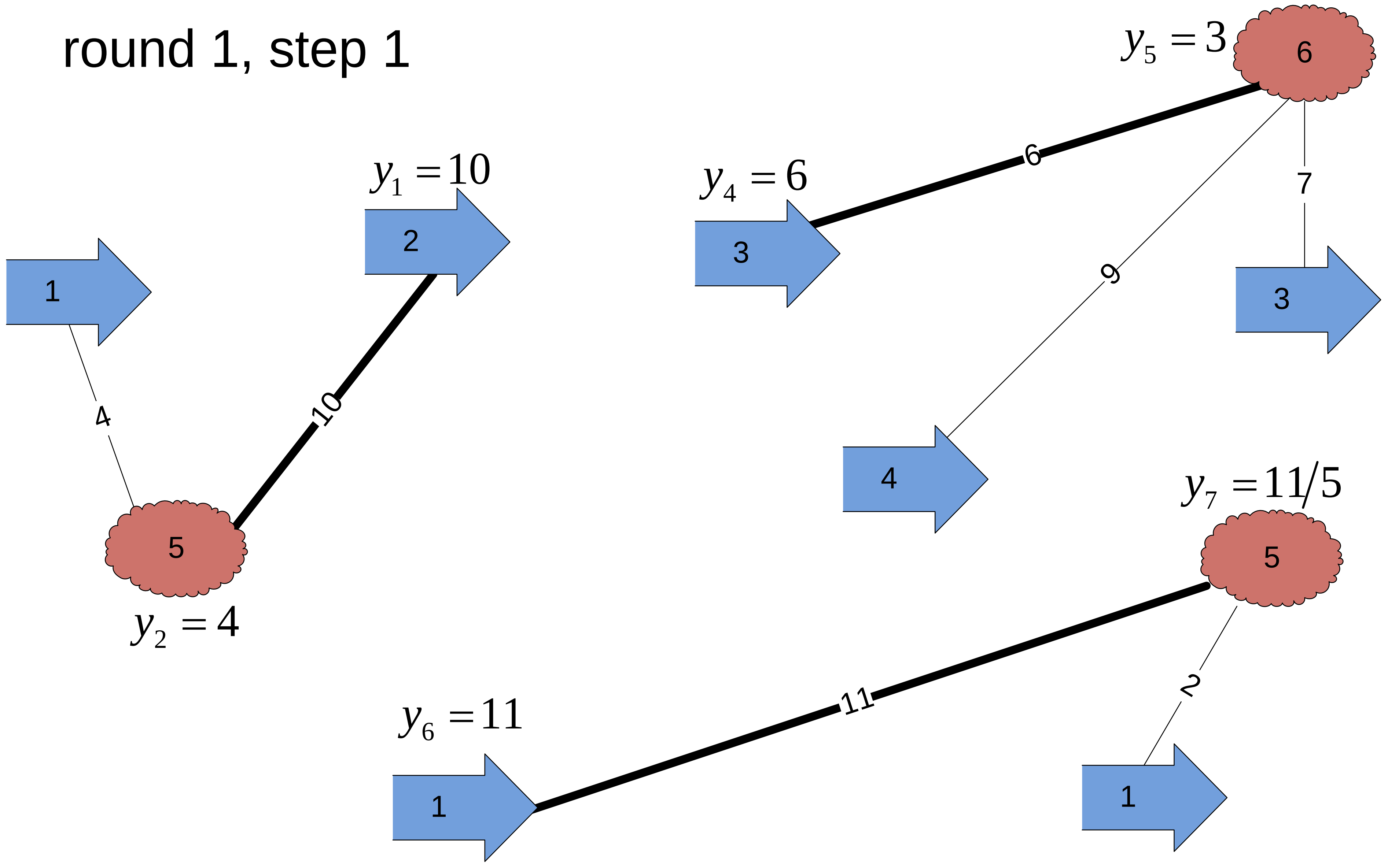


- Distributed covering:
   1. **Create stars, where the roots have weight greater than or equal to the weight of leaves.**



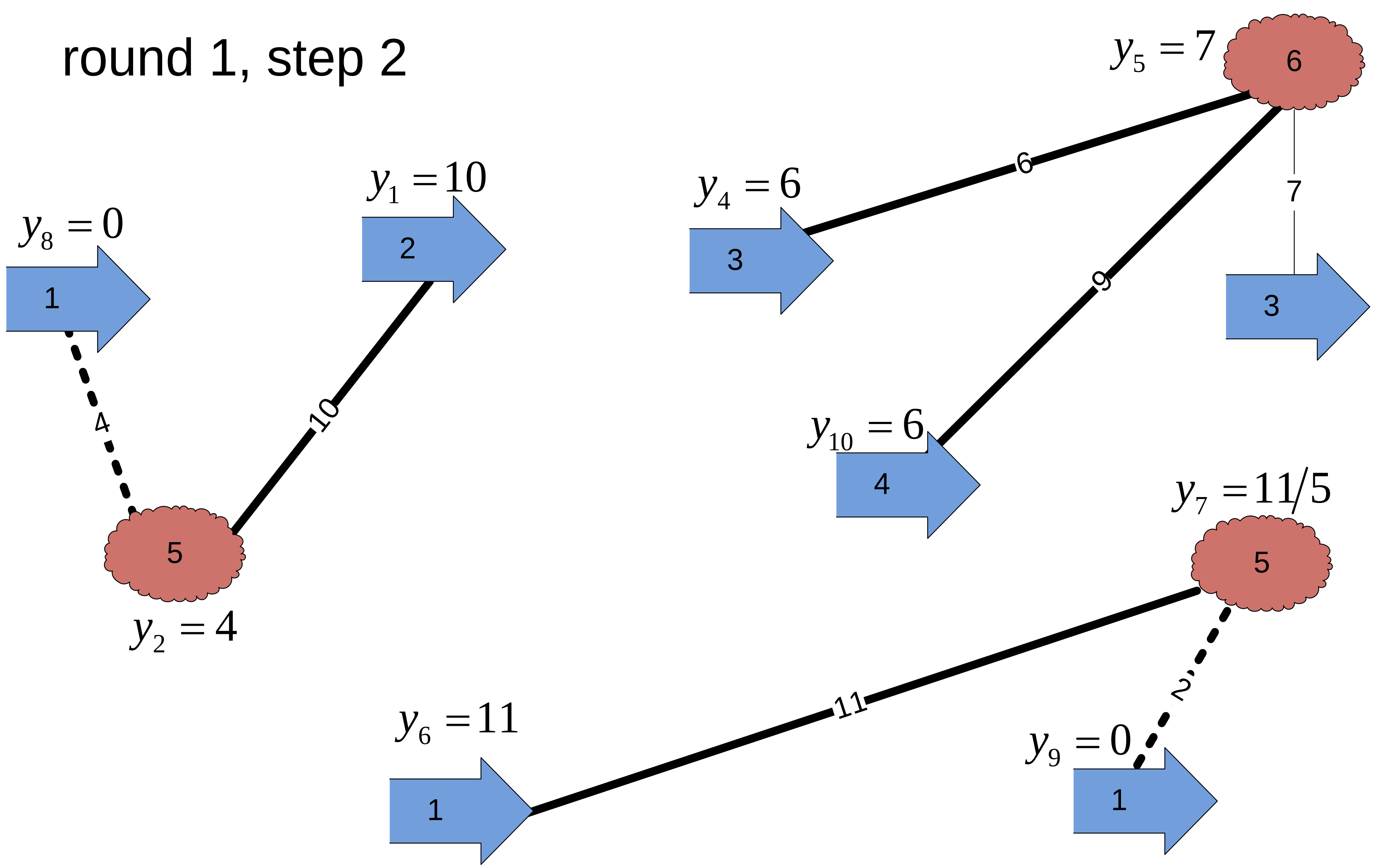
- Distributed covering:
  1. Create stars, where the roots have weight greater than or equal to the weight of leaves.
  2. **Roots perform steps to cover edges to leafs that selected them.**

round 1, step 1



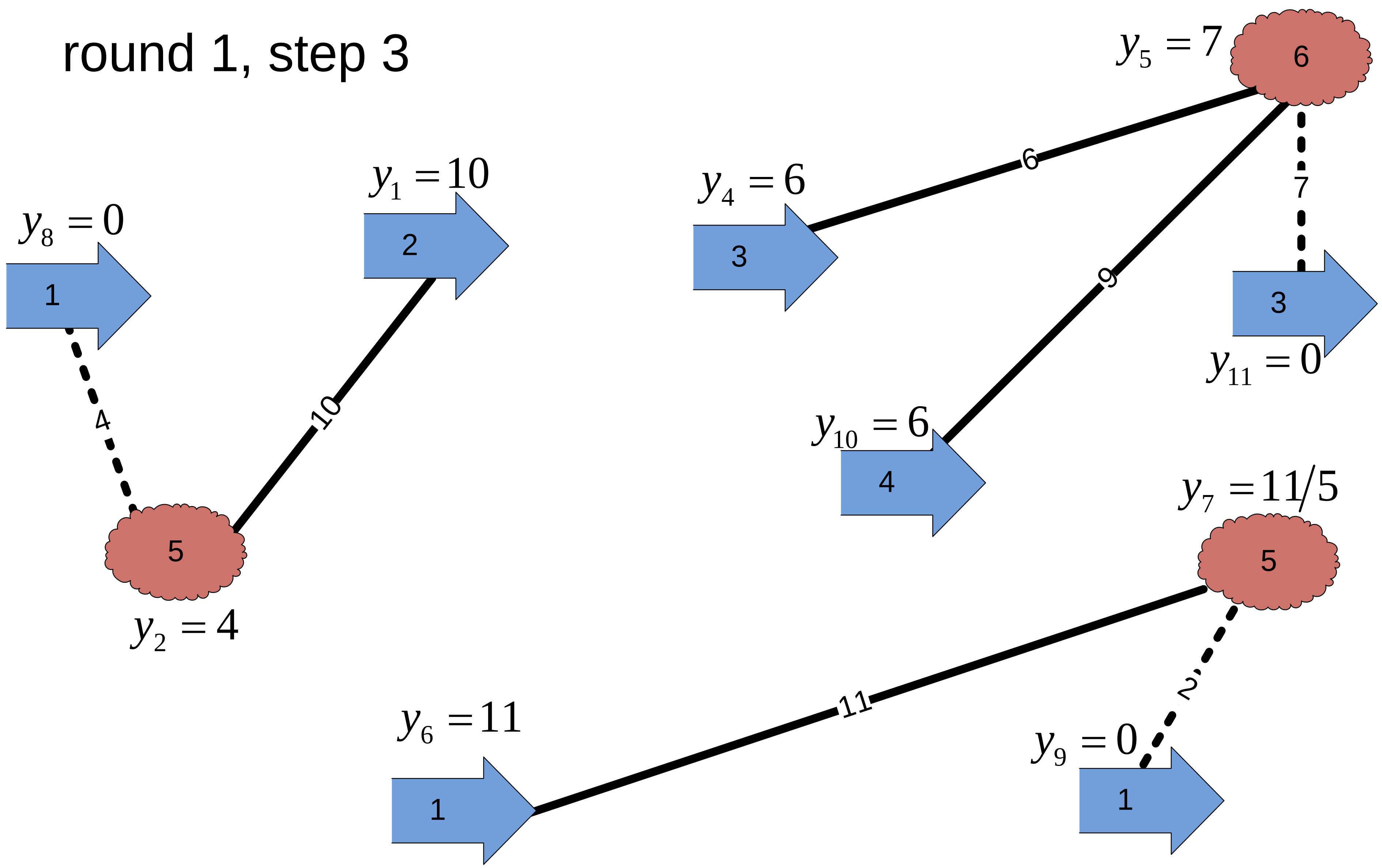
- Distributed covering:
  1. Create stars, where the roots have weight greater than or equal to the weight of leaves.
  2. **Roots perform steps to cover edges to leafs that selected them.**

round 1, step 2

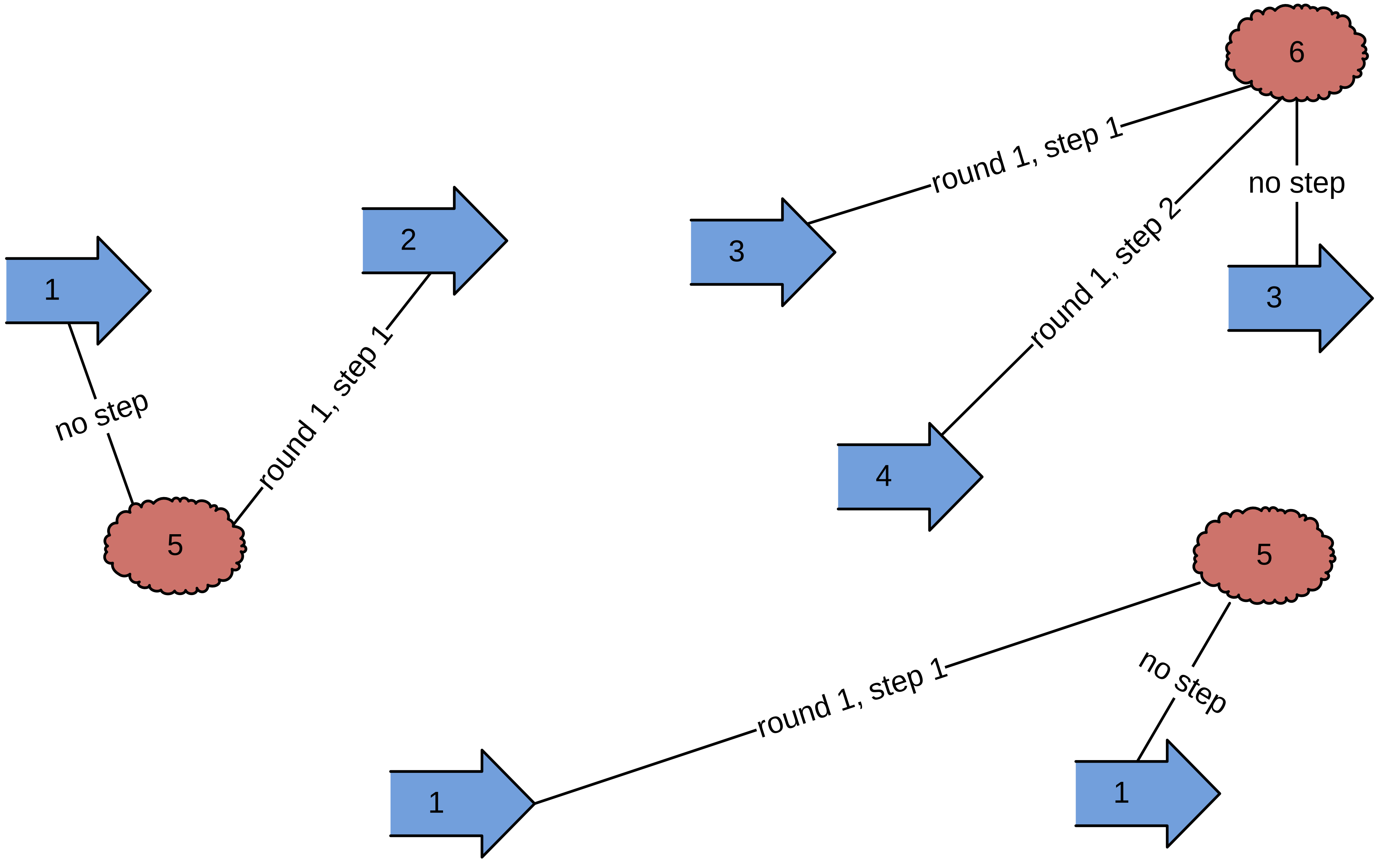


- Distributed covering:
  1. Create stars, where the roots have weight greater than or equal to the weight of leaves.
  2. **Roots perform steps to cover edges to leafs that selected them.**

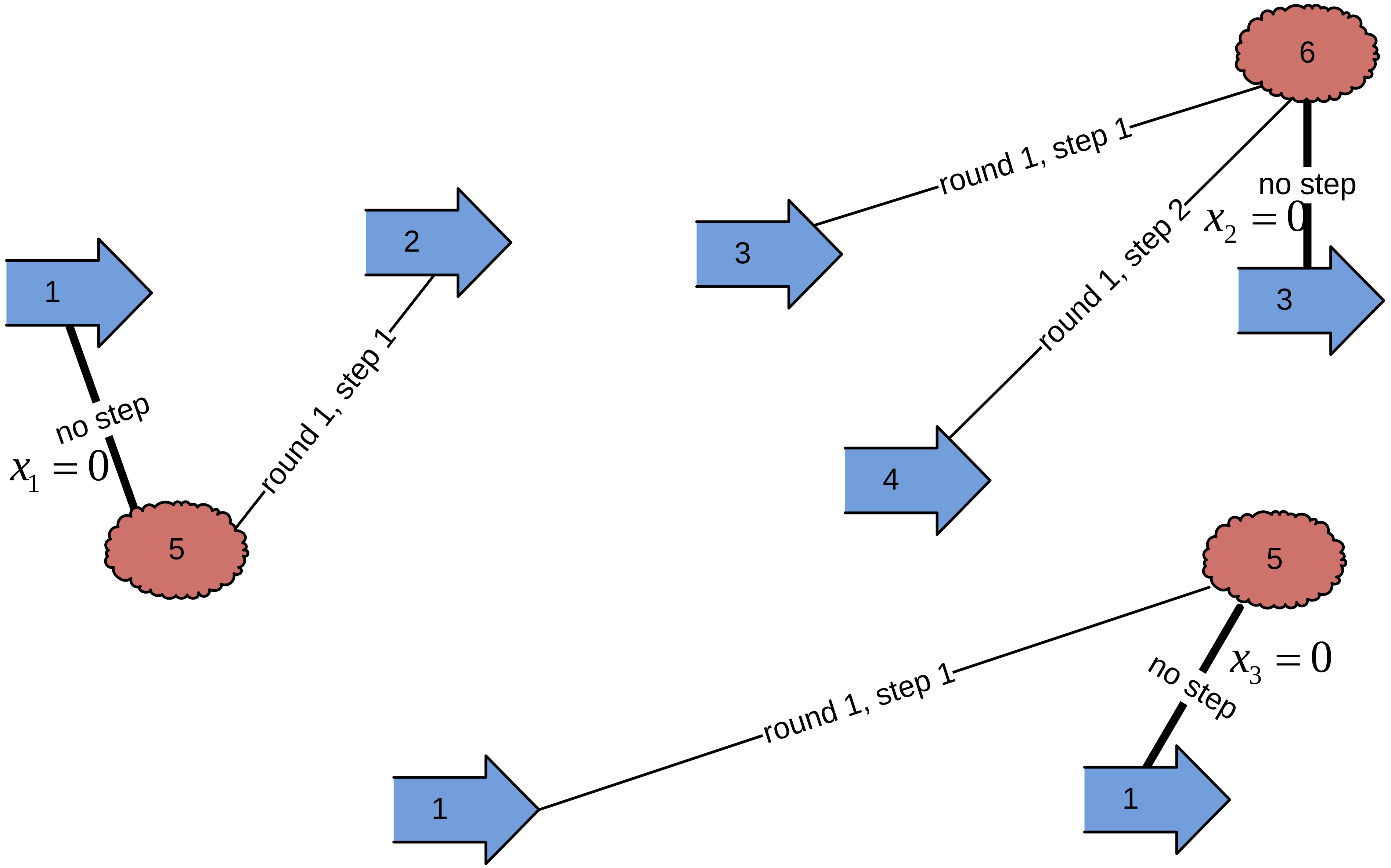
round 1, step 3



- Distributed packing:
  1. **For each edge record “when” a covering step was performed.**



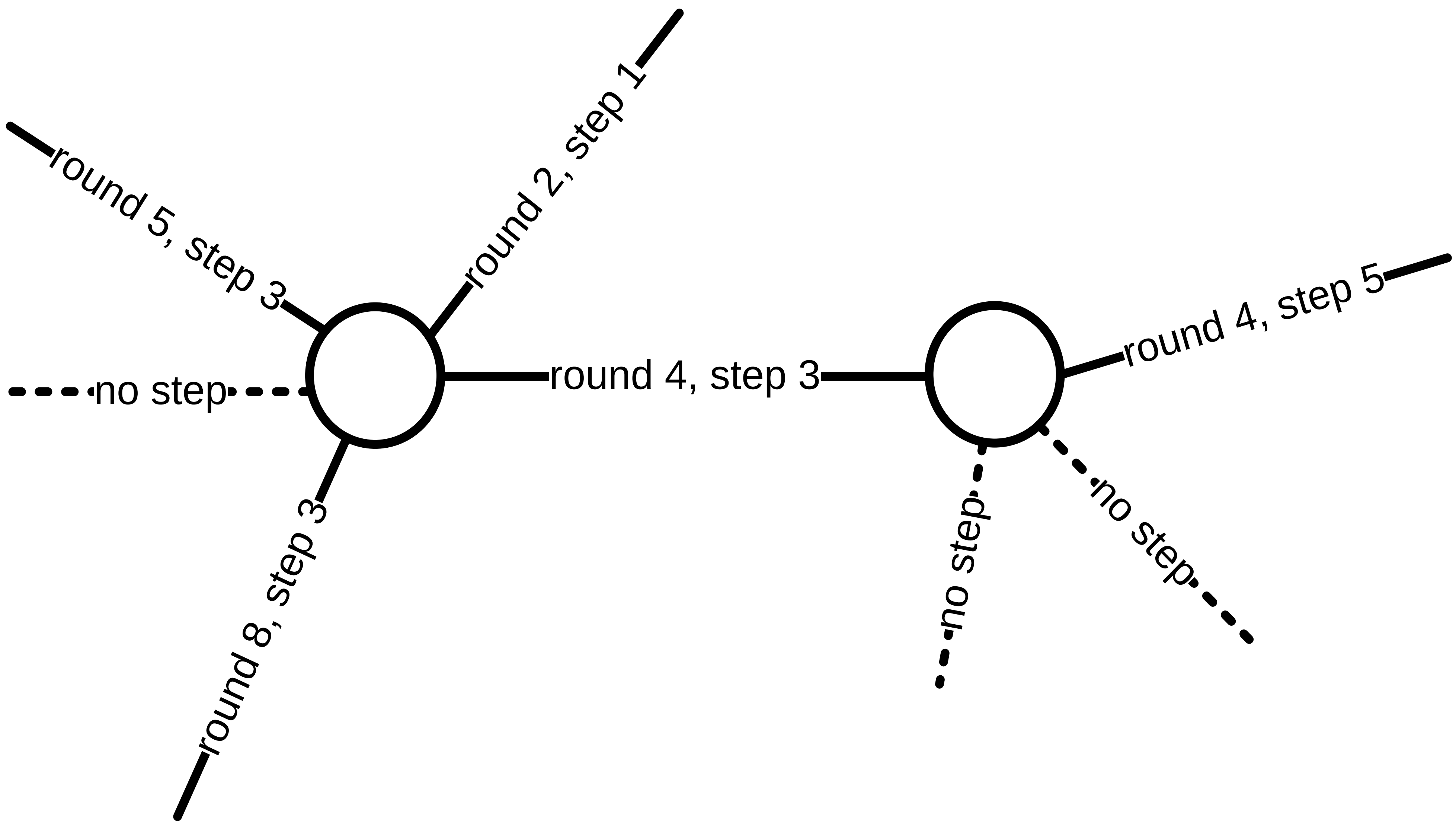
- Distributed packing:
  1. For each edge record “when” a covering step was performed.
  2. **If an edge is covered but no step was performed for it, set its packing variable to 0.**



- Distributed packing:
  1. For each edge record “when” a covering step was performed.
  2. If an edge is covered but no step was performed for it, set its packing variable to 0.
  3. **If edge *e* and all of its adjacent edges are covered:**

     a) wait until all adjacent edges that were covered later have set their packing variables.

     b) raise x_e maximally w/out violating any constraint.



- Distributed packing:
  1. For each edge record "when" a covering step was performed.
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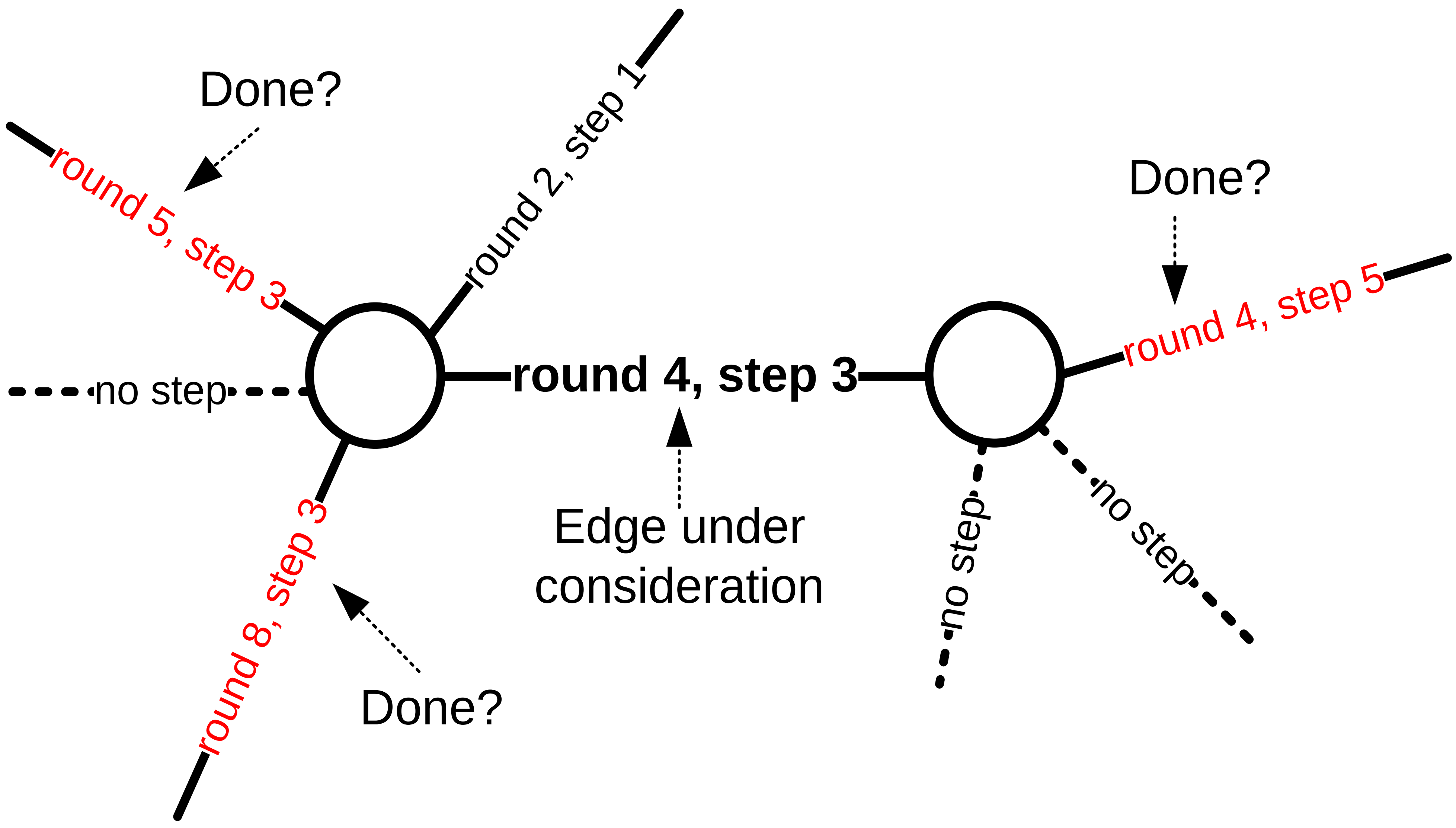

Analysis

- Guaranteed to return a 2-approximate solution, since it implements the sequential algorithm.
- What about running time?
- **Goal:** Show O(log n) rounds (w.h.p.).

Analysis of number of rounds

- **lemma:** *If the distributed covering algorithm finishes in T rounds, then the distributed packing algorithm finishes in at most $2T$ rounds.*
- **proof:** (next)

corollary: *Since $T = O(\log n)$, the distributed packing algorithm finishes in $O(\log n)$ rounds.*

Analysis of number of rounds

- Edges covered at the same round by the same root node can all set their packing variables in a single round as long as neither of them is waiting for any adjacent edge.
- Edges covered at round T (last round of covering algorithm) can immediately raise their packing variables (at round T).
- Then, edges covered at round T-1, can set their packing variables in round $T+1$.
- Edges covered at round $T-t$, can set their packing variables at round $T+t$…
- Using induction on t = 1,2,.., at most T more rounds are necessary to construct the packing solution.

Open problems

- $(1+\varepsilon)$-approximation in $O_\varepsilon(\log n)$ rounds?
- Deterministic algorithm?

thank you