PTAS for Huffman coding with unequal letter costs

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introduction

Huffman coding

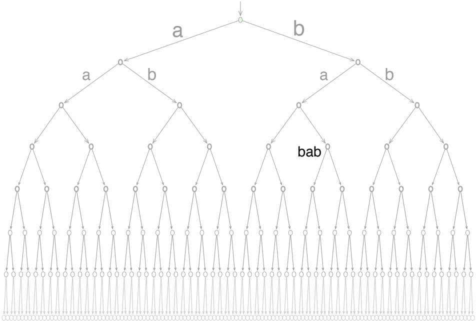
Huffman coding with unequal letter costs

A polynomial-time approximation scheme

Open questions.

Huffman coding

n frequencies
$p_1 = 4$
$p_2 = 4$
$p_3 = 2$
$p_4 = 1$
$p_5 = 1$



given: frequencies $p_1 \geq p_2 \geq \cdots \geq p_n$

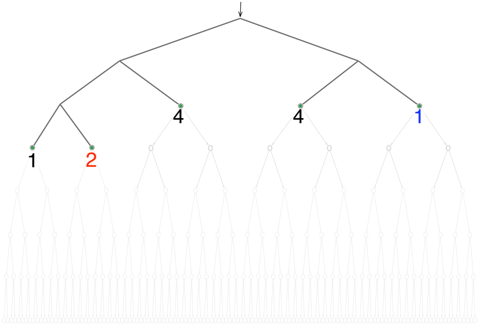
find: binary codewords $w_1, w_2, \ldots, w_n$

objective: minimize wtd average codeword length $\sum_i p_i \, |w_i|$

***prefix-free*:** no codeword is a prefix of any other codeword

A **prefix-free** code of cost 27

frequency $\rightarrow$ "word"	
4 $\rightarrow$ "ab",	cost 8
4 $\rightarrow$ "ba",	cost 8
2 $\rightarrow$ "aab",	cost 6
1 $\rightarrow$ "aaa",	cost 3
1 $\rightarrow$ "bb",	cost 2
	27



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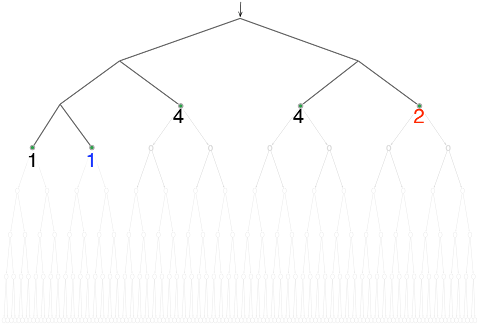
A **monotone** prefix-free code (lower cost)

$4 \longrightarrow$ "ab"
$4 \longrightarrow$ "ba"
$2 \longrightarrow$ "bb"
$1 \longrightarrow$ "aaa"
$1 \longrightarrow$ "aab"

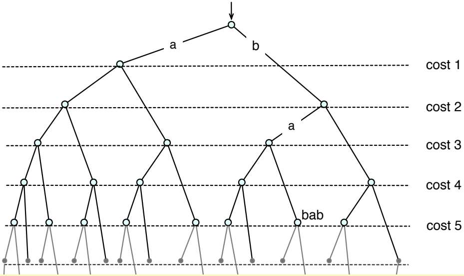


Highest frequencies are assigned to shortest codewords.

Huffman coding with **unequal letter costs**

$p_1 = 4$
$p_2 = 4$
$p_3 = 2$
$p_4 = 1$
$p_5 = 1$

each "a" costs 1
each "b" costs 2



given: letter costs $\ell_0 \leq \ell_1$ ... *in general case can have more than two letters*
frequencies $p_1 \geq p_2 \geq \cdots \geq p_n$

find: binary codewords $w_1, w_2, \ldots, w_n$

objective: minimize wtd average codeword cost, $\sum_i p_i cost(w_i)$

prefix-free: no codeword is a prefix of any other codeword

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NP-hard? c-approx?

PTAS (main result)

Theorem (GMY - STOC 2002)

For Huffman coding with unequal letter costs, for any fixed $\varepsilon > 0$, a $(1 + \varepsilon)$-approximate solution can be computed in time poly(n).

algorithm

1. Scale and round the letter costs.
2. Find a minimum-cost t-relaxed code c.
3. "Round" c to make it prefix free.

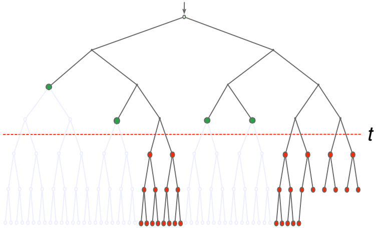
algorithm

1. Scale and round the letter costs.
2. Find a minimum-cost *t-relaxed* code c.
3. "Round" c to make it prefix free.

t-relaxed: words of cost $\geq t$ *can* be prefixes of other words

4 codewords cost $< t$:

31 codewords cost $\geq t$:



Lemma (lower bound on opt)

cost(optimal t-relaxed code) $\leq$ *cost(optimal prefix-free code)*

will take $t = O_\varepsilon(1)$ — a constant (dependent on ε)

algorithm

1. Scale and round the letter costs.
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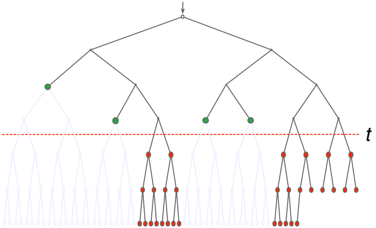
finding a minimum-cost t-relaxed code

choose words of cost $< t$
by exhaustive search

$t \approx \log(1/\varepsilon)/\varepsilon \longrightarrow$

choose words of cost $\geq t$
greedily



exhaustive search:

...for dealing with bigger-than binary alphabets

In each level $1, 2, .., t$, only *number* of codewords matters.
$\Rightarrow$ at most n^t equivalence classes of codes.
$\Rightarrow$ $n^{O(t)}$ time to search them all.

algorithm

1. Scale and round the letter costs.
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Making a t-relaxed code prefix free:

for each codeword w of cost $\geq t$:

Split w as $w = x\,y$ where $\text{cost}(x) \approx t$.

Replace w with $w' = x\,|y|\,y$, where $|y|$ is encoded in binary.

example: $w =$ aabaaababaaabbaaabbaaab

$\rightarrow$ aabaaaba1100baaabbaaabbaaab

$\rightarrow$ aabaaababbbbaaaaabbaaabbaaabbaaab

Lemma: Cost of code increases by $1 + O(\varepsilon)$ factor.

Cost of w increases by $2\log_2 \text{cost}(w)$.

Increase is at most $\varepsilon\,\text{cost}(w)$ since $\text{cost}(w) \geq t \approx \log(1/\varepsilon)/\varepsilon$.

algorithm

1. Scale and round the letter costs.
2. Find a minimum-cost t-relaxed code c.
3. "Round" c to make it prefix free.

Theorem

The cost of the code produced by the algorithm is at most $(1 + O(\varepsilon))$ times the minimum cost of any prefix-free code.

Proof.

cost(c) is at most the minimum cost of any prefix-free code.
Making c prefix-free increases its cost by a $1 + O(\varepsilon)$ factor. □

Run time: $O(n \log n) + O(f(\varepsilon) \log^2 n)$ [GMY - 2009]

Still open...

NP-hard? In P?