It would be very useful if we could simplify regular languages/expressions and determine their properties.

Algebraic Laws for languages

- $L \cup M = M \cup L$.

Union is *commutative*.

- $(L \cup M) \cup N = L \cup (M \cup N)$.

Union is *associative*.

- $(LM)N = L(MN)$.

Concatenation is *associative*

Note: Concatenation is not commutative, *i.e.*, there are L and M such that $LM \neq ML$.

- $\emptyset \cup L = L \cup \emptyset = L$.

$\emptyset$ is *identity* for union.

- $\{\epsilon\}L = L\{\epsilon\} = L$.

$\{\epsilon\}$ is *left* and *right identity* for concatenation.

- $\emptyset L = L\emptyset = \emptyset$.

$\emptyset$ is *left* and *right annihilator* for concatenation.

- $L(M \cup N) = LM \cup LN$.

Concatenation is *left distributive* over union.

- $(M \cup N)L = ML \cup NL$.

Concatenation is *right distributive* over union.

- $L \cup L = L$.

Union is *idempotent*.

- $\emptyset^* = \{\epsilon\}$, $\{\epsilon\}^* = \{\epsilon\}$.

- $L^+ = LL^* = L^*L$, $L^* = L^+ \cup \{\epsilon\}$

- $(L^*)^* = L^*$. Closure is *idempotent*

Proof:

$$w \in (L^*)^* \iff w \in \bigcup_{i=0}^{\infty} \left(\bigcup_{j=0}^{\infty} L^j \right)^i$$

$$\iff \exists k, m_1, \ldots, m_k \in \mathbb{N} : \quad w = w_1 \ldots w_k \text{ with } w_1 \text{ in } L^{m1}, \ldots, w_k \text{ in } L^{mk}$$

$$\iff \exists p \in \mathbb{N} : w \in L^p \quad \text{where } p = m_1 + \ldots + m_k$$

$$\iff w \in \bigcup_{i=0}^{\infty} L^i$$

$$\iff w \in L^* \qquad \square$$

Claim. (L ∪ M)* = (L*M*)*.

Proof. It is easy to see that L ∪ M is contained in L*M*, since L is contained in L* which is contained in L*M*, and similarly M is contained in L*M*. Thus, the LHS is contained in the RHS.

To see that the RHS is also contained in the LHS, take any w in (L*M*)*. Then, $w = w_1 w_2 \ldots w_n$, where each substring w_i is an element of L*M* and can thus be written as $x_{i1} \ldots x_{ik} y_{i1} \ldots y_{ih}$, where each sub-substring x_{ij} is an element of L and each y_{ij} an element of M. Thus, w is the concatenation of a sequence of strings, each of which is an element of L ∪ M. Therefore, it is a string in (L ∪ M)*.

The above language laws all concern regex operations and can also be written as, e.g, L + M = M + L and L(M+N) = LM + LN.

Algebraic Laws for regex's

Evidently e.g. $L((\mathbf{0}+\mathbf{1})\mathbf{1}) = L(\mathbf{01}+\mathbf{11})$

Also e.g. $L((\mathbf{00}+\mathbf{101})\mathbf{11}) = L(\mathbf{0011}+\mathbf{10111})$.

More generally

$$L((E+F)G) = L(EG+FG)$$

for any regex's E, F, and G or more generally, any languages *E*, *F*, and *G*.

- How do we verify that a general identity like above is true?

1. Prove it by hand.

2. Let the computer prove it.

In Chapter 4 we will learn how to test automatically if $E = F$, for any *concrete* regex's E and F.

We want to test *general* identities, such as $\mathcal{E} + \mathcal{F} = \mathcal{F} + \mathcal{E}$, for *any* regex's $\mathcal{E}$ and $\mathcal{F}$.
or languages

Method:

symbols

1. "Freeze" $\mathcal{E}$ to $\boldsymbol{a}_1$, and $\mathcal{F}$ to $\boldsymbol{a}_2$

2. Test automatically if the frozen identity is true, e.g. if $L(\boldsymbol{a}_1 + \boldsymbol{a}_2) = L(\boldsymbol{a}_2 + \boldsymbol{a}_1)$

Question: Does this always work?

Answer: Yes, as long as the identities use only plus, dot, and star.

i.e. reg expr of language variables

Let's denote a generalized regex, such as $(\mathcal{E}+\mathcal{F})\mathcal{E}$ by

$$\mathsf{E}(\mathcal{E},\mathcal{F})$$

Now we can for instance make the substitution $\mathbf{S}=\{\mathcal{E}/\mathbf{0},\mathcal{F}/\mathbf{11}\}$ to obtain

$$\mathbf{S}\,(\mathsf{E}(\mathcal{E},\mathcal{F}))=(\mathbf{0}+\mathbf{11})\mathbf{0}$$

Theorem 3.13: Fix a "freezing" substitution $\spadesuit = \{\mathcal{E}_1/\boldsymbol{a}_1, \mathcal{E}_2/\boldsymbol{a}_2, \ldots, \mathcal{E}_m/\boldsymbol{a}_m\}$.

Let $\mathsf{E}(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m)$ be a generalized regex. Then for any regex's $E_1, E_2, \ldots, E_m$,

or languages

$$w \in L(\mathsf{E}(E_1, E_2, \ldots, E_m))$$

if and only if there are strings $w_i \in L(E_{ji})$, s.t.

$$w = w_1 w_2 \cdots w_k$$

and

$$a_{j_1} a_{j_2} \cdots a_{j_k} \in L(\mathsf{E}(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m))$$

Or, we "think" of each regular expr variable $\mathcal{E}_i$ as a symbol a_i.

Informally, to obtain *w*, we can first pick $a_{j1}\ a_{j2} \ldots a_{jk}$ in $L(E(a_1,a_2,\ldots,a_m))$ and then substitute for each a_{ji} any string from $L(E_{ji})$.

For example, suppose $E(\varepsilon_1,\varepsilon_2) = (\varepsilon_1 + \varepsilon_2)^*$. Then string *w* is in $L((E_1+E_2)^*)$ iff $w = w_1\ w_2 \ldots w_k$ such that $a_{j1}\ a_{j2} \ldots a_{jk}$ is in $L((a_1 + a_2)^*)$ and w_i is in $L(E_{ji})$.

For example: Suppose the alphabet is $\{1, 2\}$. Let $\mathsf{E}(\mathcal{E}_1, \mathcal{E}_2)$ be $(\mathcal{E}_1 + \mathcal{E}_2)\mathcal{E}_1$, and let E_1 be $\mathbf{1}$, and E_2 be $\mathbf{2}$. Then

$$w \in L(\mathsf{E}(E_1, E_2)) = L((E_1 + E_2)E_1) =$$

$$(\{1\} \cup \{2\})\{1\} = \{11\,,\ 21\}$$

if and only if

$$\exists w_1 \in L(E_{j1}) \quad , \exists w_2 \in L(E_{j2}) \quad : \ w = w_1 w_2$$

and

$$a_{j_1} a_{j_2} \in L(\mathsf{E}(\boldsymbol{a}_1, \boldsymbol{a}_2))) = L((\boldsymbol{a}_1 + \boldsymbol{a}_2)\boldsymbol{a}_1) = \{a_1 a_1, a_2 a_1\}$$

if and only if

$j_1 = j_2 = 1$, or $j_1 = 2$, and $j_2 = 1$

In other words, w_1 is in $L(E_1)$ **U** $L(E_2) = \{1,2\}$ and w_2 is in $L(E_1) = \{2\}$.

Another example, suppose $E_1 = 1^*$ and $E_2 = 2^*$. Then
$L_0 = L((E_1 + E_2)E_1) = L((1^* + 2^*)1^*) = L(1^* + 2^*1^*)$.
$L((a_1 + a_2)a_1) = \{a_1\, a_1 + a_2\, a_1\}$.

String w is in L_0 iff there exist w_1 in $L(E_{j1})$ and w_2 in $L(E_{j2})$ such that $w = w_1\, w_2$ and $a_{j1}\, a_{j2}$ is in $\{a_1\, a_1 + a_2\, a_1\}$.

See page 120 of the textbook.

Proof of Theorem 3.13: We do a structural induction of E.

Basis: If E $= \epsilon$, the frozen expression is also ϵ.

If E $= \emptyset$, the frozen expression is also $\emptyset$.

If E $= \mathcal{E}_1$, the frozen expression is $\boldsymbol{a}_1$. Now

$w \in L(\mathrm{E}(E_1))$ if and only if
w is in $L(E_1)$, since $L(\mathrm{E}(\boldsymbol{a}_1)) = \{\boldsymbol{a}_1\}$.

Induction:

Case 1: E = F + G.

Then ♠(E) = ♠(F) + ♠(G), and
L(♠(E)) = L(♠(F)) ∪ L(♠(G))

concrete or languages

Let F' and and G' be regex's. Then $w \in L(F' + G')$ if and only if $w \in L(F')$ or $w \in L(G')$.
Also, a string u is in E($\boldsymbol{a}_1, ..., \boldsymbol{a}_m$) iff it is in F($\boldsymbol{a}_1, ..., \boldsymbol{a}_m$) or in G($\boldsymbol{a}_1, ..., \boldsymbol{a}_m$). See the book for the rest of the proof using the I.H.

Case 2: E = F.G.

Then ♠(E) = ♠(F).♠(G), and
L(♠(E)) = L(♠(F)).L(♠(G))

concrete or languages

Let F' and and G' be regex's. Then $w \in L(F'.G')$ if and only if $w = w_1 w_2$, $w_1 \in L(F')$ and $w_2 \in L(G')$.
Also, a string u is in E($\boldsymbol{a}_1, ..., \boldsymbol{a}_m$) iff u = u_1u_2 where u_1 is in F($\boldsymbol{a}_1, ..., \boldsymbol{a}_m$) and u_2 is in G($\boldsymbol{a}_1, ..., \boldsymbol{a}_m$). The rest is similar to the above case.

Case 3: E = F*.

Prove this case at home.

The test wouldn't work if the operation intersection were included in the regular expressions. E.g. consider $\mathcal{E} \wedge \mathcal{F} = \phi$.

The test for regular expressions and languages

Examples:

To prove $(\mathcal{L}+\mathcal{M})^* = (\mathcal{L}^*\mathcal{M}^*)^*$ it is enough to determine if $(\boldsymbol{a}_1+\boldsymbol{a}_2)^*$ is equivalent to $(\boldsymbol{a}_1^*\boldsymbol{a}_2^*)^*$

To verify $\mathcal{L}^* = \mathcal{L}^*\mathcal{L}^*$ test if $\boldsymbol{a}_1^*$ is equivalent to $\boldsymbol{a}_1^*\boldsymbol{a}_1^*$.

Question: Does $\mathcal{L}+\mathcal{M}\mathcal{L} = (\mathcal{L}+\mathcal{M})\mathcal{L}$ hold?

To prove $(a_1 + a_2)^*$ == $(a_1^* a_2^*)^*$, we first notice that, $L((a_1^* a_2^*)^*)$ is a subset of $L((a_1 + a_2)^*)$.

Since $L(a_1 + a_2)$ is a subset of $L(a_1^* a_2^*)$, $L((a_1 + a_2)^*)$ is a subset of $L((a_1^* a_2^*)^*)$.

Does $a + ba = (a + b)a$ hold?

Theorem 3.14: $\mathsf{E}(\mathcal{E}_1, \ldots, \mathcal{E}_m) = \mathsf{F}(\mathcal{E}_1, \ldots, \mathcal{E}_m) \Leftrightarrow L(\spadesuit(\mathsf{E})) = L(\spadesuit(\mathsf{F}))$

Proof:

(Only if direction) $\mathsf{E}(\mathcal{E}_1, \ldots, \mathcal{E}_m) = \mathsf{F}(\mathcal{E}_1, \ldots, \mathcal{E}_m)$ means that $L(\mathsf{E}(E_1, \ldots, E_m)) = L(\mathsf{F}(E_1, \ldots, E_m))$ for any concrete regex's $E_1, \ldots, E_m$. In particular then $L(\spadesuit(\mathsf{E})) = L(\spadesuit(\mathsf{F}))$

or languages

or languages

(If direction) Let $E_1, \ldots, E_m$ be concrete regex's. Suppose $L(\spadesuit(\mathsf{E})) = L(\spadesuit(\mathsf{F}))$. Then by Theorem 3.13,

$$w \in L(\mathsf{E}(E_1, \ldots E_m)) \Leftrightarrow$$

$$\exists w_i \in L(E_i), w = w_{j_1} \cdots w_{j_m}, a_{j_1} \cdots a_{j_m} \in L(\spadesuit(\mathsf{E})) \Leftrightarrow$$

$$\exists w_i \in L(E_i), w = w_{j_1} \cdots w_{j_m}, a_{j_1} \cdots a_{j_m} \in L(\spadesuit(\mathsf{F})) \Leftrightarrow$$

$$w \in L(\mathsf{F}(E_1, \ldots E_m))$$

See page 121 of the textbook.

Properties of Regular Languages

- *Pumping Lemma.* Every regular language satisfies the pumping lemma. If somebody presents you with fake regular language, use the pumping lemma to show a contradiction.

- *Closure properties.* Building automata from components through operations, e.g. given L and M we can build an automaton for $L \cap M$.

- *Decision properties.* Computational analysis of automata, e.g. are two automata equivalent.

- *Minimization techniques.* We can save money since we can build smaller machines.

The Pumping Lemma Informally

Suppose $L_{01} = \{0^n1^n : n \geq 1\}$ were regular.

Then it would be recognized by some DFA A, with, say, k states.

Let A read 0^k. On the way it will travel as follows:

ϵ	p_0
0	p_1
00	p_2
...	...
0^k	p_k

$\Rightarrow \exists i < j : p_i = p_j$ Call this state q.

Now you can fool A:

If $\widehat{\delta}(q, 1^i) \in F$ the machine will foolishly accept 0^j1^i.

If $\widehat{\delta}(q, 1^i) \notin F$ the machine will foolishly reject 0^i1^i.

Therefore L_{01} cannot be regular.

- Let's generalize the above reasoning.

Theorem 4.1.

The Pumping Lemma for Regular Languages.

Let L be regular.

for some strings
x, *y* and *z*

Then $\exists n, \forall w \in L : |w| \geq n \Rightarrow w = xyz$ such that

1. $y \neq \epsilon$

2. $|xy| \leq n$

3. $\forall k \geq 0,\ xy^k z \in L$

Proof: Suppose L is regular

Then L is recognized by some DFA A with, say, n states.

Let $w = a_1 a_2 \ldots a_m \in L,\ m >= n$.

Let $p_i = \hat{\delta}(q_0, a_1 a_2 \cdots a_i)$.

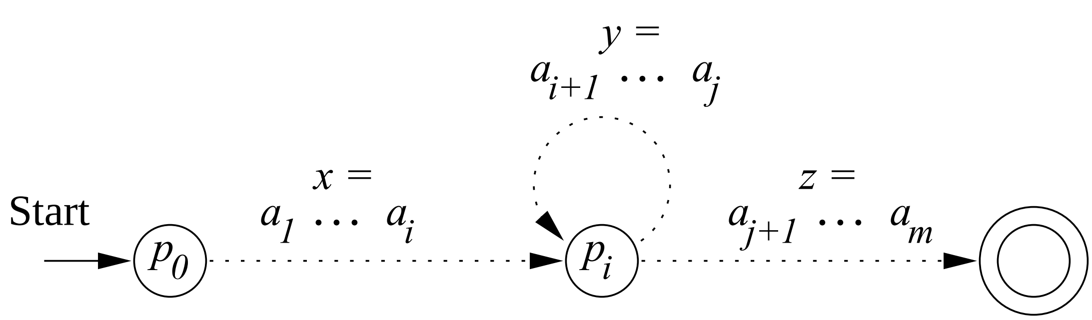
$\Rightarrow \exists i < j : p_i = p_j,\quad j <= n$

Now $w = xyz$, where

1. $x = a_1 a_2 \cdots a_i$

2. $y = a_{i+1} a_{i+2} \cdots a_j$

3. $z = a_{j+1} a_{j+2} \ldots a_m$



Evidently $xy^k z \in L$, for any $k \geq 0$. $Q.E.D.$

Example: Let L_{eq} be the language of strings with equal number of zero's and one's.

Suppose L_{eq} is regular. Then $w = 0^n 1^n \in L$.

for some x,y,z

By the pumping lemma $w = xyz$, $|xy| \leq n$, $y \neq \epsilon$ and $xy^k z \in L_{eq}$

$$w = \underbrace{000\cdots}_{x}\underbrace{\cdots 0}_{y}\underbrace{0111\cdots 11}_{z}$$

In particular, $xz \in L_{eq}$, but xz has fewer 0's than 1's.

L = {$0^i 1^j$ | i > j}
Consider string w = $0^{n+1} 1^n$.
By the pumping lemma, we can partition w as w = xyz such that |xy| <= n, y <> ε, and $xy^k z$ in L.
But xz = $0^{n+1 - |y|} 1^n$ is not in L.

Suppose $L_{pr} = \{1^p : p \text{ is prime }\}$ were regular.

Let n be given by the pumping lemma.

Choose a prime $p \geq n + 2$.

$$w = \overbrace{\underbrace{111 \cdots\cdots}_{x} \underbrace{\cdots 1}_{\substack{y \\ |y|=m}} \underbrace{1111 \cdots 11}_{z}}^{p}$$

Now $xy^{p-m}z \in L_{pr}$

$|xy^{p-m}z| = |xz| + (p - m)|y| =$
$p - m + (p - m)m = (1 + m)(p - m)$
which is not prime unless one of the factors is 1.

- $y \neq \epsilon \Rightarrow 1 + m > 1$

- $m = |y| \leq |xy| \leq n, \;\; p \geq n + 2$
$\Rightarrow p - m \geq n + 2 - n = 2.$