Solving Linear Recurrence Relations
(Last update: Sun Feb 4 14:54:50 PST 2018)

1 Tiles and Gold

Consider the following problem. We have two types of tiles, 1×1 tiles and 2×1 tiles. We want to use those tiles to cover an $n \times 1$ strip. Tiles cannot overlap and must cover exactly the strip. In how many ways this can be done? Figure 1 shows all five tilings for $n = 4$.

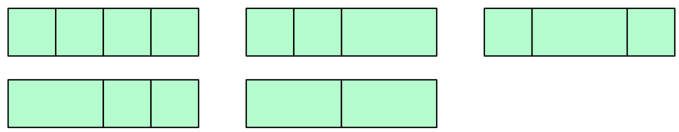

Figure 1: Five tilings for $n = 4$.

Denote by t_n the number of such tilings. For $n = 0$ we have $t_0 = 1$, because $n = 0$ represents an empty strip that requires no tiles, so there is one tiling, namely the tiling that has no tiles at all. For $n = 1$ we have $t_1 = 1$.

What about large n? Here, we reason as follows. Consider two types of such tilings: Type-1 tilings ending with a 1×1 tile, and Type-2 tilings ending with a 2×1 tile. The sets of tilings of Type-1 and Type-2 are disjoint and together they include all tiles, so t_n will be equal to the number of tilings of Type-1 plus the number of tilings of Type-2. We have t_{n-1} tilings of Type 1, because in those tilings the last unit square is occupied by a 1×1 tile but the remaining part, of length $n - 1$, can be tiled in t_{n-1} ways. Similarly, we have t_{n-2} tilings of Type 2. Putting it all together, we have

$$\begin{aligned} t_n &= t_{n-1} + t_{n-2}, \quad \text{for } n \geq 2 \\ t_0 &= 1 \\ t_1 &= 1 \end{aligned}$$

This type of formulas are called *recurrence equations* or *recurrence relations*, since each t_n is expressed in terms of earlier elements of the sequence. A recurrence equation like this uniquely determines the value of a specific t_n: we first get t_2 from t_0 and t_1, then t_3 from t_1 and t_2, and so on:

$$\begin{aligned} t_2 &= t_1 + t_0 = 1 + 1 = 2 \\ t_3 &= t_2 + t_1 = 2 + 1 = 3 \\ t_4 &= t_3 + t_2 = 3 + 2 = 5 \\ t_5 &= \ldots \end{aligned}$$

We would like, however, to have a closed-form, explicit formula for this function, that would allow us to compute each t_n independently of the previous values of the sequence.

In this particular case, you probably recognized what these numbers are: $t_n = F_n$, the n-th Fibonacci number. We know that F_n grows exponentially with n (we proved it is between $\frac{1}{2}1.5^n$ and 2^n), but we have not derived an exact formula. Let's try to do it now. First, rewrite our recurrence using the notation for

Fibonacci numbers:

$$
\begin{aligned}
F_n &= F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2 \\
F_0 &= 1 \\
F_1 &= 1
\end{aligned}
$$

What we'll do is to make an intelligent guess for how F_n looks like. Since F_n grows exponentially, we will assume that F_n is equal x^n for some unknown x, and then we will try to figure out whether there is an x for which x^n satisfies the recurrence. Plugging $F_n = x^n$ into the recurrence, we get $x^n = x^{n-1} + x^{n-2}$, which after dividing by x^{n-2} gives us the following equation for x:

$$x^2 - x - 1 \quad = \quad 0.$$

The polynomial $x^2 - x - 1$ is called the *characteristic polynomial* and the above equation is called the *characteristic equation* of the recurrence. This equation has two roots, $x_1 = \frac{1}{2}(1+\sqrt{5})$ and $x_2 = \frac{1}{2}(1-\sqrt{5})$. These numbers should look familiar: x_1 is the golden ratio $\phi \approx 1.618$ and $x_2 = 1 - \phi \approx -0.618$. What happens if we try $F_n = x_1^n$? It works for $n = 0$, but not for $n = 1$. The same problem with $F_n = x_2^n$. So both functions work for the main recurrence, but they do not satisfy the initial conditions.

So our guess was not quite right. But perhaps we can fix it. We first make the following observation. if $g(n)$ satisfies the recurrence for Fibonacci numbers (that is, $g(n) = g(n-1) + g(n-2)$) then for any constant α, the function $\alpha g(n)$ satisfies this recurrence as well. This is straightforward: just multiply $g(n) = g(n-1) + g(n-2)$ by α.

This means that all functions $\alpha_1 x_1^n$ and $\alpha_2 x_2^n$ satisfy our recurrence, except possibly for initial conditions. However, for $\alpha_1 x_1^n$ to satisfy the initial conditions we would need

$$
\begin{aligned}
\alpha_1 x_1^0 &= 1 \\
\alpha_1 x_1^1 &= 1
\end{aligned}
$$

which is impossible. We run into the same problem for $\alpha_2 x_2^n$.

But perhaps we are on the right track. What we have done so far is to identify a class of possible candidate solutions, namely all functions of the form $\alpha_1 x_1^n$ and $\alpha_2 x_2^n$, for arbitrary α_1 and α_2. All of them satisfy the main recurrence, but not the initial conditions. Perhaps the reason we cannot make it work is that we have not yet identified *all* candidate solutions?

Indeed, the following observation says that there are more: if $g_1(n)$ and $g_2(n)$ are solutions of the recurrence (not necessarily satisfying the initial condition) then so is their sum, $g(n) = g_1(n) + g_2(n)$. This is also quite easy to see, by writing down the two recurrences and adding them together. From this, we have that each function

$$F_n = \alpha_1 x_1^n + \alpha_2 x_2^n$$

satisfies the recurrence. This form is called the *general solution*. Note that now the set of all candidate solutions is parametrized by *two* parameters. So we are in good shape: if we plug this form into the initial conditions, we will get two equations for the two parameters, which in general should give us a solution.

If we do that, the equations for the initial conditions will be

$$
\begin{aligned}
\alpha_1 x_1^0 + \alpha_2 x_2^0 &= 1 \\
\alpha_1 x_1^1 + \alpha_2 x_2^1 &= 1
\end{aligned}
$$

which, after substituting, give us equations

$$\begin{aligned}
\alpha_1 + \alpha_2 &= 1 \\
\alpha_1 \cdot \tfrac{1}{2}(1+\sqrt{5}) + \alpha_2 \cdot \tfrac{1}{2}(1-\sqrt{5}) &= 1
\end{aligned}$$

So $\alpha_2 = 1 - \alpha_1$ and, after plugging this into the second equation and solving, we get

$$\begin{aligned}
\alpha_1 &= \frac{\sqrt{5}+1}{2\sqrt{5}} \\
\alpha_2 &= \frac{\sqrt{5}-1}{2\sqrt{5}}
\end{aligned}$$

This gives a solution for F_n

$$F_n = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n,$$

which can be further simplified to

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right].$$

As a last observation, note that the second term vanishes when $n \to \infty$, so for large n the value of F_n can be approximated by

$$F_n \approx \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2} \right)^{n+1}.$$

In other words, ignoring the minor perturbations caused by the second term, F_n is essentially the geometric sequence whose ratio is our old friend, the golden ratio. Oh, and if you do not feel awe at this point, you need to read this section again.

2 Life is not so Easy

Suppose now that we want to solve the recurrence

$$\begin{aligned}
f(n) &= 4f(n-1) - 4f(n-2) \\
f(0) &= 1 \\
f(1) &= 3
\end{aligned}$$

Let's follow the method we so cleverly designed in the previous solution: set up the characteristic equation, solve it for the roots, form two "elementary" solutions, then form the general solution as a linear combination of the elementary solutions, and compute its coefficients using the initial conditions.

Okie dokie, so we plug in $f(n) = x^n$, and after canceling x^{n-2} we get the characteristic equation:

$$x^2 - 4x + 4 = 0.$$

Factoring, this equation can be written as $(x-2)^2 = 0$, so (unlike before) it has only one root, $x = 2$. This is called a "double root", because 2 appears twice in the factored polynomial. (We also say that this root has multiplicity two.)

Oops, it looks that we are in trouble. We know that all functions $\alpha_1 2^n$ are good candidates for the solution, but in order to satisfy both initial conditions, we need our set of candidate solutions to be parametrized by *two* parameters. Somehow we need to find solutions of some different form. It turns out that in this case we can try the function $n2^n$. (The intuition behind this is a bit tricky, so we will not get into this.) Indeed, if we plug it in, to verify whether this is a solution, we get

$$n2^n = 4(n-1)2^{n-1} - 4(n-2)2^{n-2}.$$

and with a bit of simple algebra, it turns out that this is indeed true. As we observed before, since $n2^n$ is a solution, so is any function $\alpha_2 n2^n$. We can also take the sum of the two types of solutions. This gives us the following general form of a solution:

$$f(n) = \alpha_1 2^n + \alpha_2 n2^n.$$

From here, it's downhill. We proceed as before, setting up the equations for α_1, α_2 from the initial conditions:

$$\begin{aligned} \alpha_1 \cdot 2^0 + \alpha_2 \cdot 0 \cdot 2^0 &= 1 \\ \alpha_1 \cdot 2^1 + \alpha_2 \cdot 1 \cdot 2^1 &= 3 \end{aligned}$$

which reduces to

$$\begin{aligned} \alpha_1 &= 1 \\ 2\alpha_1 + 2\alpha_2 &= 3 \end{aligned}$$

giving us $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$. Therefore the final solution is

$$f(n) = 2^n + \tfrac{1}{2} n2^n.$$

To double check, note that we indeed have $f(0) = 1$ and $f(1) = 3$. To be on the safe side, we should also check the value for $n = 2$. Then the recurrence relation gives us $f(2) = 4f(1) - 4f(0) = 12 - 4 = 8$. From the above formula, we get $2^2 + \frac{1}{2} \cdot 2 \cdot 2^2 = 8$, so we are good, yippie!

3 Homogeneous Recurrence Equations

The two examples involve what we call *linear homogeneous recurrence equations (with constant coefficients).* They have degree 2, because each value depends on two previous values and their characteristic equations are quadratic equations that we all know how to solve (right?). Our derivation of the solutions can be generalized to solve any linear homogeneous equation, and we present this method in this section.

In general, linear homogeneous recurrence equations have the following form:

$$\begin{aligned} f_n &= a_1 f_{n-1} + a_2 f_{n-2} + \ldots + a_d f_{n-d} \\ f_0 &= b_0 \\ f_1 &= b_1 \\ &\ldots \\ f_{d-1} &= b_{d-1} \end{aligned}$$

where $a_1, \ldots, a_d$ are some real-valued coefficients (we assume that $a_d \neq 0$), and $b_0, \ldots, b_{d-1}$ are real numbers, the initial d elements of the sequence $\{f_n\}$. The parameter d is called the degree of the recurrence. The equations $f_i = b_i$ are called the *initial conditions*. (We tacitly assume that the recurrence in the first line is defined only for $n \geq d$.)

We can write it in a more compact form:

$$\begin{aligned} f_n &= \sum_{i=1}^{d} a_i f_{n-i} \quad n \geq d \\ f_n &= b_n, \qquad\quad n = 0, 1, \ldots, d-1 \end{aligned}$$

Solution method. The idea is to first find d different solutions $g_1(n), g_2(n), ..., g_d(n)$ of our recurrence, that we will call *elementary solutions*. Then any linear combination $\alpha_1 g_1(n) + ... + \alpha_d g_d(n)$ is also a solution, where $\alpha_1, ..., \alpha_d$ are some arbitrary, unknown coefficients. We can then use the initial conditions to compute these coefficients. Specifically, we proceed as follows:

(h1) Solve the characteristic equation:

$$x^d = a_1 x^{d-1} + a_2 x^{d-2} + ... + a_d.$$

Let $r_1, ..., r_k$ be the roots of this equation, and denote by μ_i the multiplicity of r_i, for each $i = 1, ..., k$. (The multiplicity of r_i, if you don't remember, is the number of times the factor $(x - r_i)$ appears in the factorization of the polynomial.) Recall that we always have $\mu_1 + \mu_2 + ... + \mu_k = d$, that is the sum of the multiplicities equals the degree of the polynomial.

(h2) Form a list of d elementary solutions. To achieve this, for each root r_i, add functions

$$r_i^n, n r_i^n, n^2 r_i^n, ..., n^{\mu_i - 1} r_i^n$$

to the list of elementary solutions.

(h3) Denote by $g_1(n), ..., g_d(n)$ the list of elementary solutions obtained above. The general solution of the recurrence is $f_n = \sum_{i=1}^{d} \alpha_i g_i(n)$.

(h4) Use the initial conditions to set up a system of d equations $\sum_{i=1}^{d} \alpha_i g_i(n) = b_n$, for $n = 0, 1, ..., d-1$. Solve this system of equations to compute the values of $\alpha_1, ..., \alpha_d$.

Example. Let's solve

$$\begin{aligned} f_n &= f_{n-1} + 6 f_{n-2} \\ f_0 &= 1 \\ f_1 &= 8 \end{aligned}$$

The characteristic equation is $x^2 = x + 6$. The roots are $r_1 = 3$ and $r_2 = -2$ (both with multiplicity one). Thus the general solution form is $f_n = \alpha_1 3^n + \alpha_2 (-2)^n$.

Plugging it into the initial conditions, we get equations

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1 \\ 3\alpha_1 - 2\alpha_2 &= 8 \end{aligned}$$

Solving, we obtain $\alpha_1 = 2$, $\alpha_2 = -1$, and thus the final solution is $f_n = 2 \cdot 3^n - (-2)^n$.

To verify the solution, we compute its values for $n = 0, 1, 2$. For $n = 0$, we get $2 \cdot 3^0 - (-2)^0 = 1$ and for $n = 1$ we get $2 \cdot 3^1 - (-2)^1 = 8$. Both values agree with the initial conditions. For $n = 2$, our formula gives $2 \cdot 3^2 - (-2)^2 = 14$, which agrees with $f_2 = f_1 + 6 f_0 = 8 + 6 = 14$.

Example. Let's now solve a recurrence of degree 3:

$$\begin{aligned} f_n &= 7f_{n-1} - 16f_{n-2} + 12f_{n-3} \\ f_0 &= 0 \\ f_1 &= 0 \\ f_2 &= 1 \end{aligned}$$

The characteristic equation is $x^3 - 7x^2 + 16x - 12 = 0$. The polynomial $x^3 - 7x^2 + 16x - 12$ can be factored as $(x-2)^2(x-3)$, which means that it has one root $r_1 = 2$ with multiplicity 2 and another root $r_2 = 3$ with multiplicity 1. Thus the general solution is

$$f_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 3^n.$$

We now set up the equations corresponding to the initial conditions ($n = 0, 1, 2$):

$$\begin{aligned} \alpha_1 + \alpha_3 &= 0 \\ 2\alpha_1 + 2\alpha_2 + 3\alpha_3 &= 0 \\ 4\alpha_1 + 8\alpha_2 + 9\alpha_3 &= 1 \end{aligned}$$

We solve this system, getting $\alpha_1 = -1$, $\alpha_2 = -\frac{1}{2}$ and $\alpha_3 = 1$. Thus the final solution is

$$f_n = -2^n - \tfrac{1}{2}n2^n + 3^n.$$

To verify, we have $f_0 = -2^0 - \frac{1}{2} \cdot 0 \cdot 2^0 + 3^0 = 0$, $f_1 = -2^1 - \frac{1}{2} \cdot 1 \cdot 2^1 + 3^1 = 0$, $f_2 = -2^2 - \frac{1}{2} \cdot 2 \cdot 2^2 + 3^2 = 1$, and $f_3 = -2^3 - \frac{1}{2} \cdot 3 \cdot 2^3 + 3^3 = 7$, where the last value agrees with that computed from the recurrence: $f_3 = 7f_2 - 16f_1 + 12f_0 = 7$.

4 Correctness (sketch)

We derived our solution method working on examples. Can we actually prove that our method always works? We covered all ingredients of the proof already in these examples, so let us now put it all together into a more coherent argument. The math needed to prove correctness for arbitrary degrees is beyond the scope of this class, so we will only work out the proof for equations of degree 2, that is recurrence equations of the form:

$$\begin{aligned} &f_n = a_1 f_{n-1} + a_2 f_{n-2} \\ &f_0 = b_0 \\ &f_1 = b_1 \end{aligned}$$

Recall that we assume that $a_2 \neq 0$, since otherwise the degree of this equation would not be 2.

Two questions that one needs to answer first, when solving equations, are:

- Is there a solution?
- If there is one, is it unique?

The nice thing about recurrence equations is that they always have a unique solution. Why? This follows directly from the equation: f_0 and f_1 are given in the initial conditions, and any other value f_n is computed from the previous two values. So the solution exists and is unique.

We thus need to show that our method does produce a valid solution. We will break the proof into two parts. In the first part we will argue that our formula for the general solution is correct, and in the second part we will show that we can always find parameters α_1 and α_2 for which the initial conditions are satisfied.

So let's now prove correctness of the general form. In other words, for now we ignore the initial conditions, we are only interested in solutions of the main recurrence.

Recall that the characteristic equation is

$$x^2 - a_1 x - a_2 = 0$$

Lemma 1 *(i) If r is a root of the characteristic equation then r^n is a solution of the recurrence.*

(ii) If r is a double root of the characteristic equation then nr^n is a solution of the recurrence.

Corollary 1 *(i) If the characteristic equation has two different roots r_1, r_2, then $g_1(n) = r_1^n$ and $g_2(n) = r_2^n$ are solutions of the recurrence.*

(ii) If the characteristic equation has a double root r then $g_1(n) = r^n$ and $g_2(n) = nr^n$ are solutions of the recurrence.

Lemma 2 *If $g_1(n)$ and $g_2(n)$ are solutions of the recurrence, then for any α_1, α_2, their linear combination $g(n) = \alpha_1 g_1(n) + \alpha_2 g_2(n)$ is also a solution.*

The last lemma implies that our general solution form, namely $f_n = \alpha_1 g_1(n) + \alpha_2 g_2(n)$ is indeed correct. That completes the first part of the proof. We still need to show that for some α_1 and α_2 the initial conditions will be satisfied.

Lemma 3 *There are numbers α_1 and α_2 for which the initial conditions are satisfied, that is*

$$\begin{aligned} \alpha_1 g_1(0) + \alpha_2 g_2(0) &= b_0 \\ \alpha_1 g_1(1) + \alpha_2 g_2(1) &= b_1 \end{aligned}$$

This lemma completes the correctness proof.

5 Non-Homogeneous Recurrences

Example. We'll start with an example. Consider the following recurrence equation:

$$\begin{aligned} f(n) &= 4f(n-1) - 4f(n-2) + 3n \\ f(0) &= 0 \\ f(1) &= 2 \end{aligned}$$

This looks a lot like a recurrence we solved before, except the additional term $3n$. This additional term is called the *non-homogeneous* term and the recurrence of this type is called a *non-homogeneous linear recurrence equation.*

Let's try guessing first. Rather than guessing a complete general form of a solution, let us try to guess just one particular function that will satisfy the recursive formula (but not necessarily the initial conditions). The reasoning here is simple: since the non-homogeneous term is $3n$, a linear function, it is reasonable to expect that some linear function of n would satisfy the recursive formula. So we try a function $f''(n) = \beta_1 n + \beta_2$, where β_1 and β_2 are some unknown numbers, that we will need to find. Plugging it into the recursive formula, we get

$$\beta_1 n + \beta_2 \quad = \quad 4[\beta_1(n-1) + \beta_2] - 4[\beta_1(n-2) + \beta_2] + 3n$$

This looks kind of messy at first, but it's actually very simple, since a lot of things cancel. After simplification, we get

$$(\beta_1 - 3)n + (\beta_2 - 4\beta_1) \quad = \quad 0.$$

Now we need to scratch our heads for a while. We have one equation and two unknowns, so how we can possibly find β_1 and β_2? But it is not true that we have one equation. The above equation must hold *for every* n, meaning that it's really infinitely many equations, one for each value of $n = 0, 1, 2, ...$ One way to solve it is to use $n = 0$ first, which gives $\beta_2 - 4\beta_1 = 0$, and then $n = 1$, which will give $\beta_1 - 3 = 0$ (using the fact that $\beta_2 - 4\beta_1 = 0$). So we end up with equations

$$\begin{aligned} \beta_2 - 4\beta_1 &= 0 \\ \beta_1 - 3 &= 0 \end{aligned}$$

that give us $\beta_1 = 3$ and $\beta_2 = 12$. Another way to think about this argument is that the formula $(\beta_1 - 3)n + (\beta_2 - 4\beta_1)$ represents a straight line, and the equation $(\beta_1 - 3)n + (\beta_2 - 4\beta_1) = 0$ says that this line is identical to the zero-line (the x-axis), so both coefficient must be 0, giving us the equations above.

We can conclude that the function $f''(n) = 3n + 12$ satisfies the recursive formula $f(n) = 4f(n-1) - 4f(n-2) + 3n$. That's neat, but so what? It does not satisfy the initial conditions, and it's not clear why this would be useful for anything.

Well, let's look next at the homogeneous equation associated with our recurrence: $f(n) = 4f(n-1) - 4f(n-2)$. We know (see the previous section) that $f'(n) = \alpha_1 2^n + \alpha_2 n 2^n$ satisfies this recurrence, for any choice of α_1 and α_2. So we have

$$\begin{aligned} f'(n) &= 4f'(n-1) - 4f'(n-2) \\ f''(n) &= 4f''(n-1) - 4f''(n-2) + 3n \end{aligned}$$

and if we add these two equations, we get

$$[f'(n) + f''(n)] \quad = \quad 4[f'(n) + f''(n-1)] - 4[f'(n) + f''(n-2)] + 3n.$$

This means that the function $f(n) = f'(n) + f''(n) = \alpha_1 2^n + \alpha_2 n 2^n + 3n + 12$ satisfies our recursive formula. It has two parameters, so we can use it as a general solution, just like before, and compute α_1 and α_2 from the initial conditions:

$$\begin{aligned} \alpha_1 2^0 + \alpha_2 \cdot 0 \cdot 2^0 + 3 \cdot 0 + 12 &= 0 \\ \alpha_1 2^1 + \alpha_2 \cdot 1 \cdot 2^1 + 3 \cdot 1 + 12 &= 2 \end{aligned}$$

giving $\alpha_1 = -12$ and $\alpha_2 = 11/2$. Thus the final solution is

$$f(n) \quad = \quad -12 \cdot 2^n + \tfrac{11}{2} \cdot n 2^n + 3n + 12.$$

Verifying this for $n = 0, 1, 2$ we get $f(0) = 0$, $f(1) = 2$ and $f(2) = 14$, which agrees with the recurrence equation.

Solving non-homogeneous recurrences. Non-homogeneous linear recurrences (with constant coefficients) have the following form:

$$\begin{aligned} f_n &= \sum_{i=1}^{k} a_i f_{n-i} + q(n) \\ f_i &= b_i, \quad i = 0, 1, \ldots, d-1 \end{aligned}$$

where $a_1, \ldots, a_d$ are some real-valued coefficients (we assume that $a_d \neq 0$), $b_0, ..., b_{d-1}$ are real numbers, the initial d elements of the sequence $\{f_n\}$, and $q(n)$ is some function independent of the sequence $\{f_n\}$.

The associated homogeneous recurrence is obtained by dropping the inhomogeneous term $q(n)$:

$$f_n = \sum_{i=1}^{d} a_i f_{n-i}$$

Solution method. Overall, the strategy is similar to the homogeneous case: determine first the general form of a solution, and then use the initial conditions to compute the parameters α_i. To obtain the general solution, as it turns out, all we need to do is to find some (any) particular solution and add it to the general solution of the associated homogeneous equation.

In more detail, we proceed as follows (following notation from the book):

(nh1) Consider the associated homogeneous equation $f_n = \sum_{i=1}^{d} a_i f_{n-i}$. Determine the roots of the characteristic polynomial and the general form of the solution, with some parameters α_i (as in the previous section). Denote this general solution by f'_n.

(nh2) Determine an *arbitrary* solution to the non-homogeneous equation. We call it a *particular* solution and denote it f''_n. (See below for how to find such particular solutions.)

(nh3) The general solution to the non-homogeneous equation is $f_n = f'_n + f''_n$. As before, this solution will be parametrized by some unknown constants α_i.

(nh4) As in the solution for the homogeneous case, use the initial conditions to set up the appropriate equations and determine the parameters α_i.

Finding a particular solution. We still need to figure out how to do step (2) above, finding a particular solution to the original equation. The method consists, essentially, of guessing the form of a solution, parametrized by some unknown values, and then computing the values of the parameters that will make it work.

The trick is to guess $f''(n)$ to be a function of the same form as $q(n)$. For example, if $q(n) = 2^n$, we guess $f''(n) = \beta 2^n$, for some unknown constant β. We then plug it into the recurrence and see if there is a β for which $\beta 2^n$ satisfies the recurrence. If not, we try $\beta n 2^n$, then $\beta n^2 2^n$, and so on.

If $q(n)$ is a polynomial, we try a polynomial of the same degree. For example, if $q(n) = n^2 + 2n$, we try first $\beta_1 n^2 + \beta_2 n + \beta_3$, then $\beta_1 n^3 + \beta_2 n^2 + \beta_3 n$, and so on.

Example. Let's solve

$$\begin{aligned} f_n &= f_{n-1} + 6 f_{n-2} + 2 \\ f_0 &= 5 \\ f_1 &= 7 \end{aligned}$$

The associated homogeneous equation is $f_n = f_{n-1} + 6f_{n-2}$, and its general solution (as we determined earlier) is $f'_n = \alpha_1 3^n + \alpha_2(-2)^n$.

Next, we determine a particular solution of the non-homogeneous equation. We have $q(n) = 2$, so we try first a particular solution of the form $f''(n) = \beta$, a constant. Plugging it in, we get

$$\beta = \beta + 6\beta + 2$$

so $\beta = -\frac{1}{3}$. Thus the general solution is

$$f_n = \alpha_1 3^n + \alpha_2(-2)^n - \tfrac{1}{3}.$$

Using the initial conditions, we have

$$\begin{aligned} \alpha_1 3^0 + \alpha_2(-2)^0 - \tfrac{1}{3} &= 5 \\ \alpha_1 3^1 + \alpha_2(-2)^1 - \tfrac{1}{3} &= 7 \end{aligned}$$

which simplifies to

$$\begin{aligned} \alpha_1 + \alpha_2 &= \tfrac{16}{3} \\ 3\alpha_1 - 2\alpha_2 &= \tfrac{22}{3} \end{aligned}$$

We solve it, getting $\alpha_1 = \frac{18}{5}$ and $\alpha_2 = \frac{26}{15}$. So the solution is

$$f_n = \tfrac{18}{5} \cdot 3^n + \tfrac{26}{15} \cdot (-2)^n - \tfrac{1}{3}.$$

Example. Let's solve

$$\begin{aligned} f_n &= f_{n-1} + 6f_{n-2} + 5 \cdot 3^n \\ f_0 &= 1 \\ f_1 &= 4 \end{aligned}$$

The associated homogeneous equation is the same as before, so its general solution is $f'_n = \alpha_1 3^n + \alpha_2(-2)^n$.

We now look for a particular solution. The first guess is $f''(n) = \beta 3^n$. After plugging it into the recurrence, we get

$$\beta 3^n = \beta 3^{n-1} + 6\beta \cdot 3^{n-2} + 5 \cdot 3^n,$$

which, after dividing by 3^{n-2}, simplifies to $9\beta = 3\beta + 6\beta + 45$, so there is no such β.

Oh well, life is tough. Next we try $f''(n) = \beta n 3^n$. We plug it in, getting

$$\beta n 3^n = \beta(n-1)3^{n-1} + 6\beta(n-2)3^{n-2} + 5 \cdot 3^n.$$

After dividing by 3^{n-2} and simplifying, we see that all terms involving n cancel and we get $\beta = 3$. So $f''(n) = 3n \cdot 3^n$, and our general solution is

$$f_n = \alpha_1 3^n + \alpha_2(-2)^n + 3n \cdot 3^n.$$

All that remains to do is to compute α_1, α_2. From the initial conditions for $n = 0, 1$ we get equations

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1 \\ 3\alpha_1 - 2\alpha_2 + 9 &= 4 \end{aligned}$$

whose solution is $\alpha_1 = -\frac{3}{5}$ and $\alpha_2 = \frac{8}{5}$. Thus the final solution is

$$f_n \quad = \quad -\tfrac{3}{5}3^n + \tfrac{8}{5}(-2)^n + 3n \cdot 3^n.$$

To be on the safe side, it's a good idea to verify this solution. We check it for $n = 0, 1, 2$. For $n = 0, 1$, we get $f_0 = -\frac{3}{5}3^0 + \frac{8}{5}(-2)^0 + 3 \cdot 0 \cdot 3^0 = 1$, $f_1 = -\frac{3}{5}3^1 + \frac{8}{5}(-2)^1 + 3 \cdot 1 \cdot 3^1 = 4$, which agrees with the initial conditions. For $n = 2$, the recurrence gives us $f_2 = f_2 + 6f_{n-2} + 5 \cdot 3^2 = 55$, while our solution gives $f_2 = -\frac{3}{5}3^2 + \frac{8}{5}(-2)^2 + 3 \cdot 2 \cdot 3^2 = 55$ as well. So we're good.

Example. Let's solve

$$\begin{aligned}
&A_n = 6A_{n-1} - 12A_{n-2} + 8A_{n-3} + 2^n \\
&A_0 = 1 \\
&A_1 = 2 \\
&A_2 = 3
\end{aligned}$$

The inhomogeneous term is 2^n. So to find a particular solution, we start with $A''_n = \beta 2^n$. Plugging it into the recurrence we get

$$\beta 2^n = 6\beta 2^{n-1} - 12\beta 2^{n-2} + 8\beta 2^{n-3} + 2^n$$

We now divide by 2^{n-3}, which gives us

$$8\beta = 24\beta - 24\beta + 8\beta + 8$$

and this equation reduces to $0 = 8$, so it does not have a solution. The next step is try to $A''_n = \beta n 2^n$. We plug it into the recurrence:

$$\beta n 2^n = 6\beta(n-1)2^{n-1} - 12\beta(n-2)2^{n-2} + 8\beta(n-3)2^{n-3} + 2^n$$

and divide by 2^{n-3}:

$$8\beta n = 24\beta(n-1) - 24\beta(n-2) + 8\beta(n-3) + 8.$$

This last equation reduces again to $0 = 8$, so our second guess was not right. The next attempt is $A''_n = \beta n^2 2^n$. Plugging it into the recurrence, we get

$$\beta n^2 2^n = 6\beta(n-1)^2 2^{n-1} - 12\beta(n-2)^2 2^{n-2} + 8\beta(n-3)^2 2^{n-3} + 2^n$$

To simplify, we divide by 2^{n-3}:

$$8\beta n^2 = 24\beta(n-1)^2 - 24\beta(n-2)^2 + 8\beta(n-3)^2 + 8.$$

After expanding all squared expressions, this becomes

$$8\beta n^2 = 24\beta(n^2 - 2n + 1) - 24\beta(n^2 - 4n + 4) + 8\beta(n^2 - 6n + 9) + 8.$$

However, yet again, everything cancels, leaving us with $0 = 8$, so the guess $A''_n = \beta n^2 2^n$ will not work either.

Next we try $A''_n = \beta n^3 2^n$. We plug it into the recurrence, getting

$$\beta n^3 2^n = 6\beta(n-1)^3 2^{n-1} - 12\beta(n-2)^3 2^{n-2} + 8\beta(n-3)^3 2^{n-3} + 2^n$$

To simplify, we divide by 2^{n-3}:

$$8\beta n^3 = 24\beta(n-1)^3 - 24\beta(n-2)^3 + 8\beta(n-3)^3 + 8.$$

Expanding, this gives us

$$8\beta n^3 = 24\beta(n^3 - 3n^2 + 3n - 1) - 24\beta(n^3 - 6n^2 + 12n - 8) + 8\beta(n^3 - 9n^2 + 27n - 27) + 8.$$

All terms involving n cancel, and from the remaining terms we get $\beta = \frac{1}{6}$. So our particular solution is

$$A''_n = \tfrac{1}{6} n^3 2^n$$

Next, we compute the general solution of the corresponding homogeneous equation:

$$A'_n = 6A'_{n-1} - 12A'_{n-2} + 8A'_{n-3}$$

The characteristic polynomial is $x^3 - 6x^2 + 12x - 8 = 0$. The candidates for roots are $-1, 1, -2, 2, -4, 4, -8, 8$. Trying these out, we find out that $r_1 = 2$ is a root. Factoring, we get that

$$x^3 - 6x^2 + 12x - 8 = (x-2)^3,$$

so the multiplicity of $r_1 = 2$ is $\mu_1 = 3$. This gives us the general form of the solution for the homogeneous equation:

$$A'_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 n^2 2^n.$$

Combining the two solutions, we get the general form of the solution of the original inhomogeneous equation:

$$A_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 n^2 2^n + \tfrac{1}{6} n^3 2^n.$$

It now only remains to compute α_1, α_2 and α_3. From the initial conditions, we get

$$\begin{aligned} \alpha_1 &= 1 \\ 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \tfrac{1}{3} &= 2 \\ 4\alpha_1 + 8\alpha_2 + 16\alpha_3 + \tfrac{16}{3} &= 3 \end{aligned}$$

Solving this, we obtain $\alpha_1 = 1$, $\alpha_2 = \frac{11}{24}$ and $\alpha_3 = -\frac{5}{8}$. Thus the final solution is

$$A_n = 2^n + \tfrac{11}{24} n 2^n - \tfrac{5}{8} n^2 2^n + \tfrac{1}{6} n^3 2^n.$$

Example. One more example showing how to set up a recurrence equation. Let T_n be the number of strings of length n formed from symbols $0, 1, ..., 9$ in which the number of 0's is even. Thus $T_0 = 1$, because the empty string has zero 0's, an even number. $T_1 = 9$, because all strings consisting of one symbol other than 0 satisfy the condition. For $n = 2$, we get $T_2 = 2 \cdot 9 + 1 = 82$.

Next, we derive the recurrence for T_n. This can be done by considering two types of strings of length n: those ending with 0 and those ending with a symbol other than 0.

Consider first the strings ending with a symbol different than 0. If v is such a string then it can be written as ud, where d is the last digit of v, $d \neq 0$, and u has length $n-1$ and has an even number of 0's. There are 9 choices for d and T_{n-1} choices for u, which gives us $9T_{n-1}$ strings of the first type.

We now consider strings ending with 0, that is strings v that can be written as $u0$. In u the number of 0's is odd and u has $n-1$ symbols. How many such u's we have? We can compute this number by computing the number of *all* strings of length $n-1$ (and there are 10^{n-1} of them) and subtracting the number of those that have an even number of 0's, that is T_{n-1}. Thus there are $10^{n-1} - T_{n-1}$ strings of the second type.

Adding the two values, we obtain that $T_n = 9T_{n-1} + 10^{n-1} - T_{n-1} = 8T_{n-1} + 10^{n-1}$. We thus obtain the recurrence

$$\begin{aligned} T_n &= 8T_{n-1} + 10^{n-1} \\ T_0 &= 1 \end{aligned}$$

To solve it, we follow the process outlined above. We first find a particular solution of the form $T''_n = \beta 10^n$. Plugging it into the recurrence, we get

$$\beta 10^n = 8\beta 10^{n-1} + 10^{n-1},$$

which gives $\beta = \frac{1}{2}$. So $T''_n = \frac{1}{2}10^n$. Next, we solve the homogeneous version of our recurrence:

$$T_n = 8T_{n-1}.$$

The characteristic equation is $x - 8 = 0$, so the general form of the homogeneous equation is $T'_n = \alpha 8^n$.

Thus the general form of the original equation is $T_n = \alpha 8^n + \frac{1}{2}10^n$. Plugging it into the initial condition for $n = 0$:

$$T_0 = \alpha 8^0 + \tfrac{1}{2}10^0 = 1$$

we obtain $\alpha = \frac{1}{2}$. Thus the final solution is

$$T_n = \tfrac{1}{2}8^n + \tfrac{1}{2}10^n.$$