Dirac's Theorem

Recall that a Hamiltonian cycle in a graph $G = (V, E)$ is a cycle that visits each vertex exactly once. Unlike for Euler cycles, no simple characterization of graphs with Hamiltonian cycles is known. In fact, the question whether a given graph has a Hamiltonian cycle is known to be *NP-complete* – a technical term that, for all practical purposes, implies that this question cannot be solved efficiently.

Some conditions that imply the existence of Hamiltonian cycles are known though. A rather obvious intuition is that if a graph is sufficiently dense (has enough edges) then it should have long cycles. The theorem below shows that this intuition is indeed right.

Theorem 1 *(Dirac's theorem) Let $G = (V, E)$ be a graph with n vertices in which each vertex has degree at least $n/2$. Then G has a Hamiltonian cycle.*

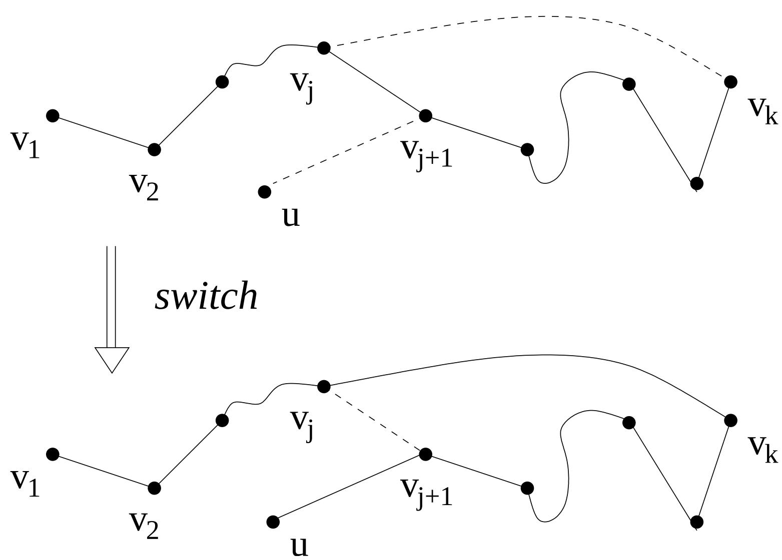
Proof: The proof is by an explicit construction, that is, we show that if G satisfies the condition in the theorem that we can construct a Hamiltonian cycle in G.

The idea is to pick some vertex v_1 arbitrarily and gradually extend a path P starting from v_1, say $P = v_1 v_2 ... v_k$, where all vertices v_j are different. Eventually, if $k = n$, P will be a Hamiltonian path.

Initially, $P = (v_1)$. Suppose that we have already constructed $P = v_1 v_2 ... v_k$. We now show that as long as $k < n$ we can always extend P.

If v_k has a neighbor $u \in V$ that is not on P, then it is easy to extend P, for we can simply append u at the end of P. In other words, we can take $v_{k+1} = u$ and the new extended path will be $v_1 v_2 ... v_k v_{k+1}$.

The second case is when all neighbors of v_k are on P. This case is a bit more tricky. The idea is this: We will show that there is a neighbor v_j of v_k such that v_{j+1} has a neighbor outside P. Then we will perform a *switch operation* that transforms P into the following path: $v_1 v_2 ... v_j v_k v_{k-1} ... v_{j+1} u$, as in the figure below:



Notice that the new path is indeed longer than P by one vertex.

It is now sufficient to prove that such vertex v_j always exists. Since all neighbors of v_k are on P and are different than v_k, we have $k - 1 \geq \deg(v_k) \geq n/2$, so $k \geq n/2 + 1$. Let's do this: for each neighbor v_j of v_k, we mark the next vertex on P, that is v_{j+1}. Since *all* neighbors of v_k are on P, this way we will mark $\deg(v_k)$ vertices.

Consider any vertex u not on P. If none of u's neighbors were marked, then by adding the numbers of u's neighbors, the marked vertices, and u itself, we would get that the total number of vertices in G is at least $\deg(u) + \deg(v_k) + 1 \geq n/2 + n/2 + 1 > n$ – a contradiction. Therefore there must be a marked vertex that is a neighbor of u. But this means, exactly, that there will be a vertex v_j as in the figure above, and the switch operation can be applied.

Summarizing what we've done so far, the above argument shows that G has a Hamiltonian path $P = v_1 v_2 ... v_n$. But the theorem actually says that G has a Hamiltonian cycle, so we are not really done yet. This is left as an exercise (for extra credit!!!) In other words, you need to show how (under thee assumptions from the theorem) you can convert P into a Hamiltonian cycle. $\square$