Line-line Intersection

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Given two straight lines, each represented by two points, there are three possible cases:

1. The two lines are parallel and disjoint. They do not intersect or they intersect in infinity.
2. The two lines are parallel and overlapping. Their intersection is a straight line equal to any of them.
3. The two lines are not parallel. They intersect in a single point.

The following derivation shows how to easily find out which case applies to the two lines and find their intersection point if they intersect in a point.

First, let us assume that the first line S_1 is represented by two points p_1 and p_2. Similarly, the second line S_2 is represented by two points p_3 and p_4. Each point p_i is represesented by two coordinates (x_i, y_i). Let us also assume that the intersection point is p_0.

We define the following vectors:

$$a = \overrightarrow{p_1p_2} = p_2 - p_1 = (x_2 - x_1, y_2 - y_1)$$

$$b = \overrightarrow{p_1p_0} = p_0 - p_1 = (x_0 - x_1, y_0 - y_1)$$

$$c = \overrightarrow{p_3p_4} = p_4 - p_3 = (x_4 - x_3, y_4 - y_3)$$

$$d = \overrightarrow{p_3p_0} = p_0 - p_3 = (x_0 - x_3, y_0 - y_3)$$

Since the intersection point p_0 is on the first line, we have

$$||a \times b|| = 0$$

$$(x_2 - x_1)(y_0 - y_1) - (x_0 - x_1)(y_2 - y_1) = 0$$

$$(x_2 - x_1)y_0 + (y_1 - y_2)x_0 = y_1x_2 - x_1y_2 \tag{1}$$

Similarly, since the intersection point p_0 is on the second line, we have

$$||c \times d|| = 0$$

$$(x_4 - x_3)(y_0 - y_3) - (x_0 - x_3)(y_4 - y_3) = 0$$

$$(x_4 - x_3)y_0 + (y_3 - y_4)x_0 = y_3x_4 - x_3y_4 \tag{2}$$

Now, to find the values of the two unknowns x_0 and y_0, we need to solve the two linear equations 1 and 2. We use Cramer's rule as follows.

$$D = \begin{vmatrix} x_2 - x_1 & y_1 - y_2 \\ x_4 - x_3 & y_3 - y_4 \end{vmatrix}$$

$$D_{y_0} = \begin{vmatrix} y_1 x_2 - x_1 y_2 & y_1 - y_2 \\ y_3 x_4 - x_3 y_4 & y_3 - y_4 \end{vmatrix}$$

$$D_{x_0} = \begin{vmatrix} x_2 - x_1 & y_1 x_2 - x_1 y_2 \\ x_4 - x_3 & y_3 x_4 - x_3 y_4 \end{vmatrix}$$

Finally, we find the unknowns $x_0 = |D_{x_0}|/|D|$ and $y_0 = |D_{y_0}/|D|$.
The three cases described above will map to the following cases:

1. If $|D| = 0$ and $|D_{x_0}| \neq 0$ and $D_{y_0} \neq 0$, it indicates that the equations have a solution at infinity which means that the two lines are disjoint and parallel (they intersect at infinity).

2. If $|D| = 0$ and $|D_{x_0}| = 0$ and $D_{y_0} = 0$, it indicates that the equations have infinite number of solutions which means that the two lines are coincident.

3. If $|D| \neq 0$ it indicates a single solution to the equations which means that the two lines intersect at a single point.

Example 1:

$$p_1 = (2,0), p_2 = (0,1), p_3 = (0,0), p_4 = (1,4)$$

$$D = \begin{vmatrix} 0-2 & 0-1 \\ 1-0 & 0-4 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 1 & -4 \end{vmatrix}$$

$$|D| = 9$$

$$D_{x_0} = \begin{vmatrix} 0-2 & 0 \cdot 0 - 2 \cdot 1 \\ 1-0 & 0 \cdot 1 - 0 \cdot 4 \end{vmatrix} = \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix}$$

$$|D_{x_0}| = 2$$

$$D_{y_0} = \begin{vmatrix} 0 \cdot 0 - 2 \cdot 1 & 0-1 \\ 0 \cdot 1 - 0 \cdot 4 & 0-4 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 0 & -4 \end{vmatrix}$$

$$|D_{y_0}| = 8$$

$$x_0 = 2/9$$

$$y_0 = 8/9$$

Example 2:

$$p_1 = (2,0), p_2 = (0,1), p_3 = (4,0), p_4 = (0,2)$$

$$D = \begin{vmatrix} 0-2 & 0-1 \\ 0-4 & 0-2 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ -4 & -2 \end{vmatrix}$$

$$|D| = 0$$

This means that the lines are parallel but we need to verify if it is case 1 or 2.

$$D_{x_0} = \begin{vmatrix} 0-2 & 0\cdot 0 - 2\cdot 1 \\ 0-4 & 0\cdot 0 - 4\cdot 2 \end{vmatrix} = \begin{vmatrix} -2 & -2 \\ -4 & -8 \end{vmatrix}$$

$$|D_{x_0}| = 8 \neq 0$$

This means that the two lines are parallel and disjoint.
Example 3:

$$p_1 = (1,1), p_2 = (2,2), p_3 = (3,3), p_4 = (4,4)$$

$$D = \begin{vmatrix} 2-1 & 1-2 \\ 4-3 & 3-4 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}$$

$$|D| = 0$$

This indicates that the two lines are parallel but we need to verify if they are overlapping or disjoint.

$$D_{x_0} = \begin{vmatrix} 2-1 & 1\cdot 2 - 1\cdot 2 \\ 4-3 & 3\cdot 4 - 3\cdot 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$D_{x_0} = 0$$

Which indicates an infinite number of solution and the two lines overlap.