Math 142-2, Project 2

Both partners' names here

Projects 2 and 3 will ultimately be concerned with how the frequency of a gene in a population will change over time and how the gene's effect on the survival and reproduction of individuals will affect the continued existence of the gene in the population. Project 2 will lay down the theory and develop a mathematical model. Project 3 will then use programming as a tool to analyze the consequences of this model. Although this project attempts to develop the necessary theory along the way, reading sections 32, 34, and 37 from the course text will make this project a lot easier.

Problem 1

Let's focus on one type of animal, such as a particular species of mice. Let $N(t)$ be the number of individuals that are alive at time t. We want to construct a model for the number of individuals $N(t+\Delta t)$ alive after time Δt. The population will change for two reasons: mice reproduce and die. Let the birth and death rates be b and d. b describes the probability per unit of time that a particular mouse will give birth; d describes the probability per unit of time that a particular mouse will die. (For example, a mouse might have a 5% chance of giving birth on any particular day, or alternatively, about 5% of the mice will give birth each day.)

(a) Express $N(t+\Delta t)$ in terms of $N(t)$, b, and d.

(b) The **growth rate** of the population is $R = \frac{N(t+\Delta t)-N(t)}{\Delta t N(t)}$. What is the growth rate for the population?

(c) If very small time incriments are considered ($\Delta t \to 0$), this model becomes an ODE. What is the differential equation?

(d) If the birth and death rates are constant, solve the ODE to determine how the population over time.

(e) As in the physical systems we have been considering, an equilibrium population is one which will not change over time. If the birth and death rates are constant, what are the equilibrium populations predicted by this model?

Your solution goes here

Problem 2

In practice, a population's rate of growth will not remain constant; an area only has resources for a finite number of individuals. In general, a larger population should have slower growth rate (or, alternatively, will suffer a higher death rate). A very simple model is to say that the growth rate has the form $R = a - cN(t)$, where $a > 0$ and $c > 0$ are constants. This leads to a differential equation $\frac{dN}{dt} = N(a - cN)$ known as the logistic equation.

(a) Show that the logistic equation has exactly one positive equilibrium value. This is called the carrying capacity K.

(b) The parameter a can be intepreted as the growth rate when $N \approx 0$. Rewrite the logistic equation in terms of the parameters a and K.

(c) What will happen to the population over time if $0 < N(0) = N_0 < K$? In particular, what can be said of $N(t)$ as $t \to \infty$?

(d) What will happen to the population over time if $N(0) = N_0 > K$? In particular, what can be said of $N(t)$ as $t \to \infty$?

Your solution goes here

Problem 3

All mammals are capable of sexual reproduction. We will consider a simple view of this, where each organism receives one set of genes from its mother and another set from its father. In this way, each organism contains two possibly slightly different copies of each gene. When this organism later reproduces, it passes on one copy of each gene at random to its offspring. We will be concerned with one particular gene, which comes in two versions (*alleles*), R and r. Since each individual contains two copies, its *genotype* may be RR, Rr, or rr (Rr is the same as rR). For example, a mouse with RR or Rr might have brown fur, while a mouse with rr might have black fur. An Rr individual may pass an R or an r to its offspring, but an RR individual possesses two copies of R and will thus pass R to each of its offspring. We will assume that it does not matter which copy was received from each parent; the two copies will be treated the same way. This simple view of genetics is called **Mendelian inheritance**.

(a) In the simple example above, is it possible for two black mice to mate and produce a brown mouse?

(b) In the simple example above, is it possible for two brown mice to mate and produce a black mouse?

Your solution goes here

Problem 4

We wish to model the change in frequency of the r and R genes through a population of this particular species in the presence of resource limitations. To do this, we make a number of assumptions.

- No mutations are occurring to this gene.
- $A(t)$ is the number of individuals with genotype RR.
- $B(t)$ is the number of individuals with genotype Rr.
- $C(t)$ is the number of individuals with genotype rr.
- $T(t) = A(t) + B(t) + C(t)$ is the total population.
- The death rates are a for RR, b for Rr, and c for rr. They do not depend on the total population, and they do not change over time.

- All invididuals reproduce at the same rate $m - nT(t)$.
- Other than their differing death rates, the three types of individuals are indistinguishable.
- Any population can be assumed to contain equal numbers of male and female mice.
- In the case that the gene is irrelevant ($a = b = c$), the total population $T(t)$ should obey a logistic equation.
- If only one allele is initially present ($A(0) = B(0) = 0$ or $B(0) = C(0) = 0$), the total population should also obey a logistic equation, and the allele that is initially absent should remain absent forever.
- The model should be symmetric with respect to the identity of r and R.

(a) How many offspring should be born over the course of a small interval of time Δt?

(b) In terms of $A(t)$, $B(t)$, and $C(t)$, what fraction r_A of offspring should be RR? What fraction r_B should be Rr? What fraction r_C should be rr?

(c) How many individuals of each type should die over this Δt?

(d) Taking into account both births and deaths, determine the new populations $A(t + \Delta t)$, $B(t + \Delta t)$, and $C(t + \Delta t)$.

(e) Propose a model (a system of ODE's) to describe the evolution of the populations $A(t)$, $B(t)$, and $C(t)$ over time.

(f) Show that your model satisfies the last three assumptions.

Your solution goes here