Math 142-1, Final

Name: ______________________________ ID: ______________

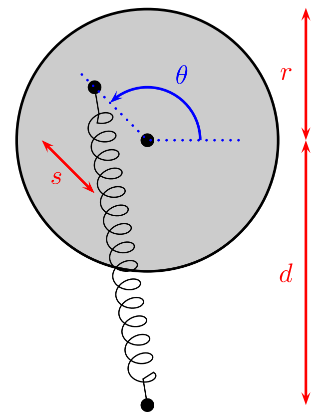
Problem 1

A 2D particle's location $\mathbf{x}$ obeys the equation of motion

$$m\ddot{\mathbf{x}} = -\frac{\mathbf{x}}{\|\mathbf{x}\|^2}.$$

Find the kinetic energy KE, potential energy ϕ, and total energy E of the system.

Problem 2



A wheel with total mass m (evenly distributed throughout) and radius r spins freely and without friction about its center. One end of a spring is attached to the wheel at a distance of s from the center of the wheel. The other end of the spring is fixed to a point d below the wheel's center. Find the equations of motion for the wheel, parameterized by the polar angle θ of the spring's attachment point. Your answer should be an ODE of the general form $\ddot{\theta} = f(\theta, \dot{\theta}, t)$. Hint: formulating energy first is easier.

Problem 3

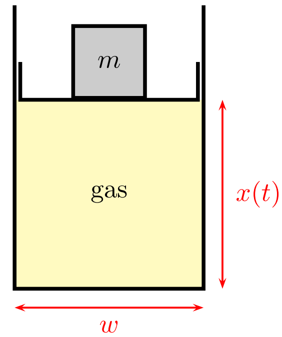
Solve the PDE for $f(x, y, t)$, subject to $f(x, y, 0) = g(x, y)$:

$$f_t(x, y, t) - y f_x(x, y, t) + x f_y(x, y, t) = 1.$$

Problem 4

Sketch the phase plane for the ODE $\ddot{x} + x(x^2+1)^{-2} = 0$. Your sketch should include representative trajectories with arrows, including trajectories through unstable equilibria (if any). Mark all stable ("$\bullet$") and unstable ("$\circ$") equilibria.

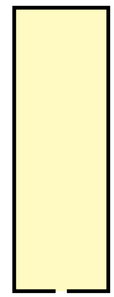
Problem 5



An idea gas is contained within a box with a moveable top. This wall is able to move up or down. Assume the walls are well-insulated. Assume that a constant fraction α of the total energy of the gas is translational energy. A mass m sits on top of the moveable ceiling, allowing it to move up and down by compressing the gas under the force of gravity, confining the gas to a $w \times w \times x(t)$ volume. Let $x(t) = x_0$ be the height of the gas volume when the mass experiences no net force. Find the equations of motion for the block. You do not need to solve them. (Hint: this related to one of the problems you did in groupwork.)

Problem 6

A light bottle is filled with gas under high pressure. A plug on the bottom is removed, opening up a hole for gas to escape. When this occurs, the bottle flies into the air like a rocket. Explain how this result would be predicted using our gas model. The stationary bottle has no momentum or energy, but after the plug is removed and the bottle goes flying, it does. You should explain where the *momentum* and *energy* come from and how they get there. You do not need to do any detailed calculations. (A simple argument like "the gas goes down, so the bottle must go up to conserve momentum" would should show that the bottle must rise, but it does not tell us *why* or *how* this occurs. The goal of this exercise is to work out what is going on at the level of the particles to provide an explanation not only for what must occur but also why and how.) Be sure your explanation correctly accounts for these observations: (1) this experiment will also work in a vacuum (2) the bottle will not go flying if the plug is not removed and (3) the temperature and energy of a gas in a uniformly translating sealed bottle does not change over time.



Problem 7

An initially piecewise constant density profile leads to two shocks and no rarefactions. Show that the shocks must eventually merge and the resulting shock moves with a velocity that is between the velocities of the original shocks. You may assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$. (If you are able to solve this problem without assuming any particular traffic following model, you will get extra credit equal in value to one problem on this exam.)

Problem 8

Determine the density of traffic on a semi-infinite road ($x \geq 0$) for all future times subject to the initial density profile $\rho(x, 0) = \frac{1}{6}\rho_{\max}$ (for $x \geq 0$) and boundary conditions $q(0, t) = \frac{2}{9}\rho_{\max}u_{\max}$ (for $t \geq 0$). Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.

Problem 9

A physical system is observed to have the following properties:

- There is exactly one equilibrium (at $x = 0$); the equilibrium is unstable
- For sufficiently large $|x|$, the system exhibits decay (energy is lost)

Construct a model (an ODE) for a 1D system which has these properties. (Hint: no linear ODE satisfies both properties; the damping coeffient can depend on x.)

Problem 10

SI units are a system of units that were chosen more or less arbitrarily to be convenient for everyday use. An alternative approach to constructing units is to select units where fundamental constants of nature become one. To fix units for distance (m), time (s), mass (kg), charge (C), and temperature (K), one must choose five fundamental constants. Planck units are a system of units designed to be purely non-arbitrary. Planck units are obtained by setting the gravitational constant (G), the reduced Planck constant ($\hbar$), the speed of light (c), the Coulomb constant (k_e), and the Boltzmann constant (k) to be one. In these units, $E = mc^2$ can be simply written as $E = m$, and the entropy of a black hole is $S = \frac{A}{4}$, where A is its area. How must the relationship between the entropy of a black and its area be written if SI units are used instead?

Quantity	Units
c	$m\,s^{-1}$
G	$kg^{-1}\,m^3\,s^{-2}$
$\hbar$	$kg\,m^2\,s^{-1}$
k_e	$kg\,m^3\,s^{-2}\,C^{-2}$
k	$kg\,m^2\,s^{-2}\,K^{-1}$
E	$kg\,m^2\,s^{-2}$
m	kg
S	$kg\,m^2\,s^{-2}\,K^{-1}$
A	m^2