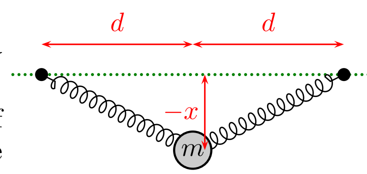
Math 142-2, Final

Name: ______________________________ ID: ______________

Problem 1

A point with mass m is attached to two fixed points $(-d, 0)$ and $(d, 0)$ by identical springs with rest length ℓ and spring constant k. Let $(0, x(t))$ be the location of the mass at any time. Find the the total energy of the system and an ODE of the form $\ddot{x} = f(x, \dot{x}, t)$ that describes the evolution of the system.



Problem 2

Sketch the phase plane for the ODE $\ddot{x} - x = 0$. Your sketch should include representative trajectories with arrows, including trajectories through unstable equilibria (if any). Mark all stable ("$\bullet$") and unstable ("$\circ$") equilibria.

Problem 3

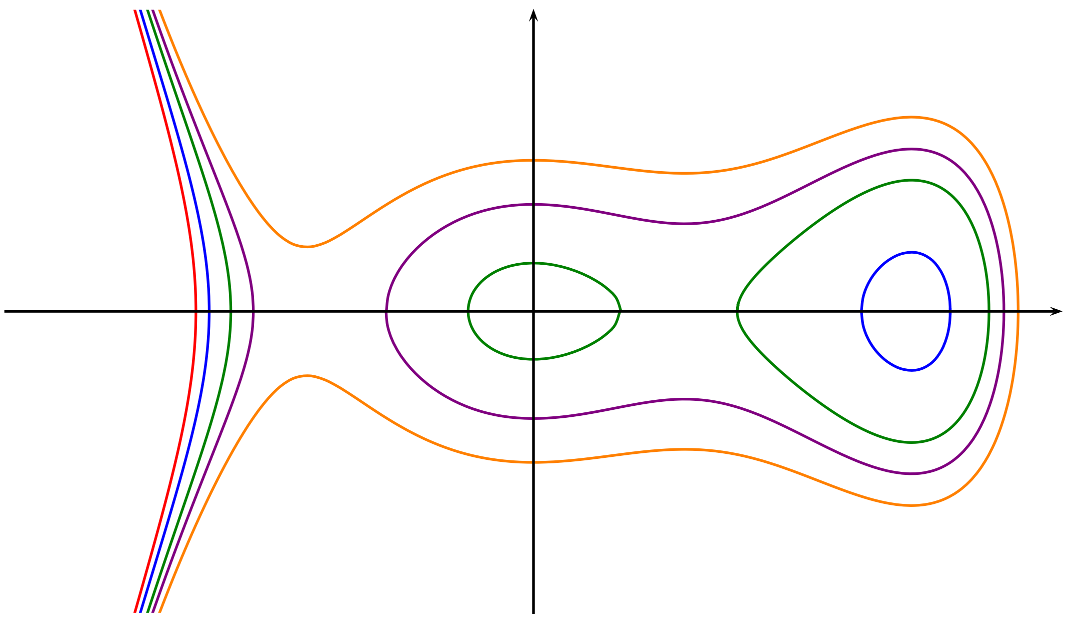
Sketch the phase plane for the ODE $\ddot{x} + x^2 = 0$. Your sketch should include representative trajectories with arrows, including trajectories through unstable equilibria (if any). Mark all stable ("$\bullet$") and unstable ("$\circ$") equilibria.

Problem 4

Sketch the phase plane for the ODE $\ddot{x}+\dot{x}=0$. Your sketch should include representative trajectories with arrows, including trajectories through unstable equilibria (if any). Mark all stable ("$\bullet$") and unstable ("$\circ$") equilibria.

Problem 5

Five energy levels for a system are shown in the phase plane below. (a) List the energy levels (red, orange, green, blue, violet) in order from lowest energy to highest energy. (b) Mark all stable ("•") and unstable ("∘") equilibria. (c) Sketch energy contours corresponding to all unstable equilibria (energy contours may contain more than one component; be sure to sketch them all). (d) Add arrows to all contours, including the ones you have added. (e) Sketch the potential energy function and show the energy levels corresponding to the five colored energy contours.



Problem 6

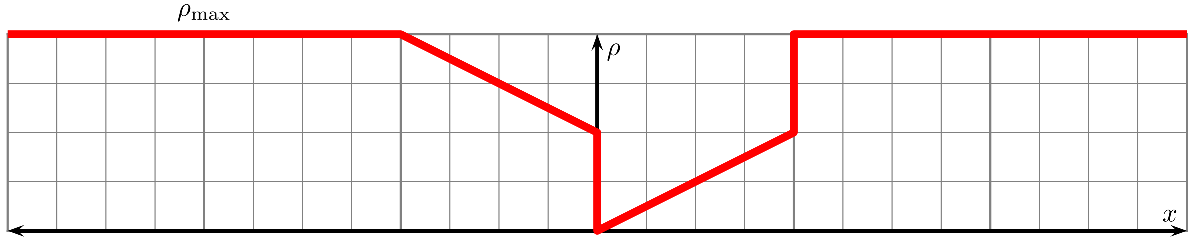
A long road has an initial uniform traffic density $\rho(x,0) = \frac{\rho}{3}$. At $t = 0$, a traffic accident occurs at $x = 0$, which effectively limits the flow rate past $x = 0$ to $q(0,t) = \frac{3}{16}u_{\max}\rho_{\max}$. Determine the traffic density for $t > 0$. Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.

Problem 7

Solve the PDE $\dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial x} + z + t = 0$ subject to $z(x, 0) = f(x)$.

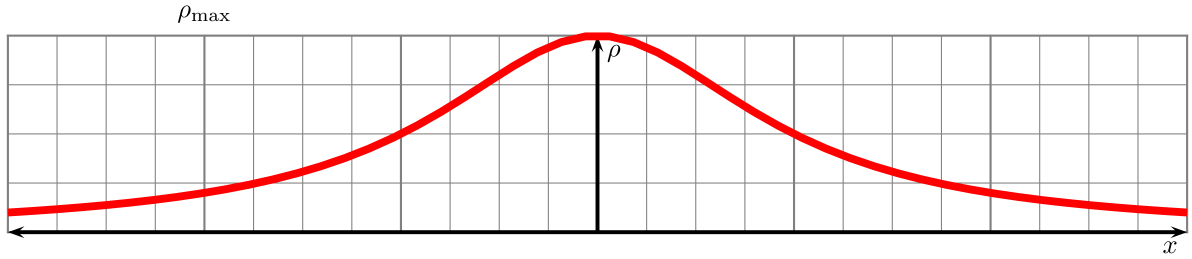
Problem 8

Identify the location of the shock ("S") and rarefaction ("R") in the initial density profile (red line). Sketch the density profile at the time when the shock and rarefaction first meet. Try to be accurate in your sketch, but do not attempt to solve the PDE analytically. Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.



Problem 9

Sketch the density distribution at the time of the first shock. Try to be accurate in your sketch, but do not attempt to solve the PDE analytically. Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.



Problem 10

The green lines in the illustration below show the locations of the shocks that occur when some piecewise constant initial density profile evolves in time. There are no rarefactions. Sketch the initial density profile and indicate which portions represent light or heavy traffic. Be sure to explain your reasoning. Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.