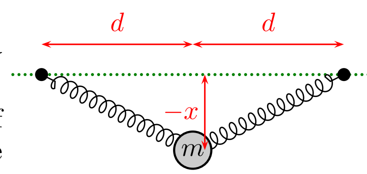
Math 142-2, Final

Solutions

Problem 1

A point with mass m is attached to two fixed points $(-d, 0)$ and $(d, 0)$ by identical springs with rest length ℓ and spring constant k. Let $(0, x(t))$ be the location of the mass at any time. Find the the total energy of the system and an ODE of the form $\ddot{x} = f(x, \dot{x}, t)$ that describes the evolution of the system.



Each spring has length $L = \sqrt{x^2 + d^2}$, displacement $\Delta x = L - \ell = \sqrt{x^2 + d^2} - \ell$, and thus potential energy $\phi_s = \frac{k}{2}\Delta x^2 = \frac{k}{2}(\sqrt{x^2 + d^2} - \ell)^2$. The total energy is

$$E = \frac{m}{2}\dot{x}^2 + k(\sqrt{x^2 + d^2} - \ell)^2 + mgx$$

The ODE is obtained by differentiation of $\dot{E} = 0$.

$$\begin{aligned}
0 &= \dot{E} \\
&= m\dot{x}\ddot{x} + 2k(\sqrt{x^2 + d^2} - \ell)\frac{d}{dt}\sqrt{x^2 + d^2} + mg\dot{x} \\
&= m\dot{x}\ddot{x} + 2k(\sqrt{x^2 + d^2} - \ell)\frac{x\dot{x}}{\sqrt{x^2 + d^2}} + mg\dot{x} \\
0 &= \ddot{x} + 2k(\sqrt{x^2 + d^2} - \ell)\frac{x}{m\sqrt{x^2 + d^2}} + g \\
\ddot{x} &= \frac{2kx}{m}\left(\frac{\ell}{\sqrt{x^2 + d^2}} - 1\right) - g
\end{aligned}$$

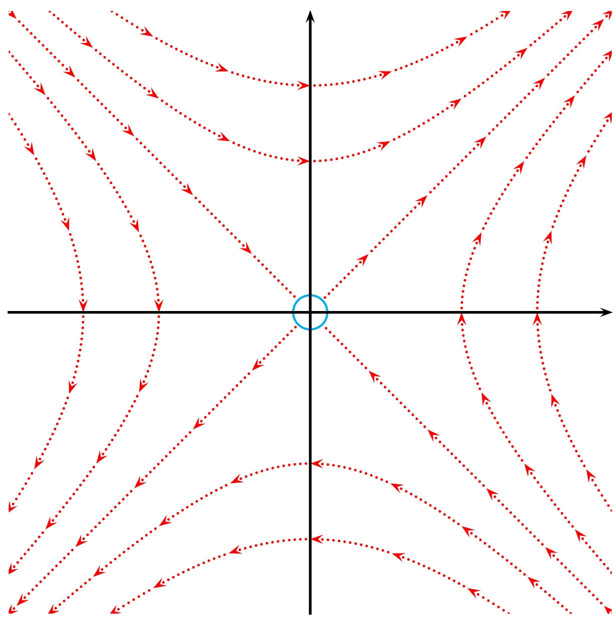
Problem 2

Sketch the phase plane for the ODE $\ddot{x} - x = 0$. Your sketch should include representative trajectories with arrows, including trajectories through unstable equilibria (if any). Mark all stable ("$\bullet$") and unstable ("$\circ$") equilibria.

Equilibria occur when $\dot{x} = \ddot{x} = 0$, so $x = 0$. This equilibrium is unstable (it corresponds to a potential energy maximum). The energy is

$$\begin{aligned} 0 &= \ddot{x} - x \\ 0 &= \dot{x}\ddot{x} - \dot{x}x \\ E &= \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 \end{aligned}$$

The energy curves are hyperbolas.



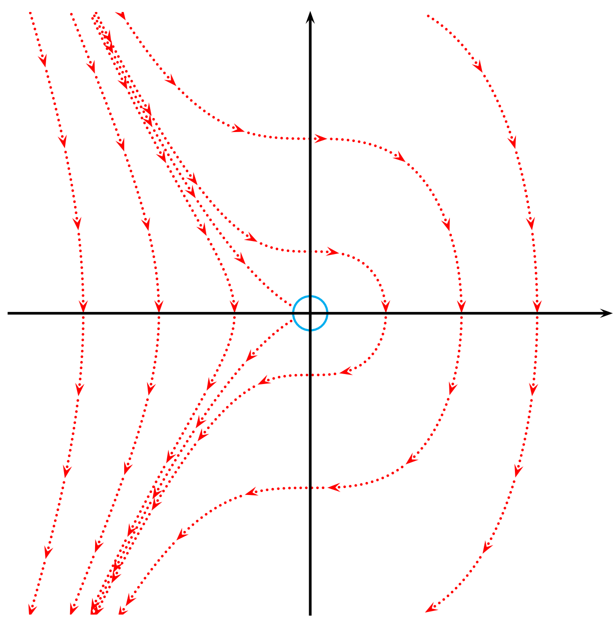
Problem 3

Sketch the phase plane for the ODE $\ddot{x} + x^2 = 0$. Your sketch should include representative trajectories with arrows, including trajectories through unstable equilibria (if any). Mark all stable ("$\bullet$") and unstable ("$\circ$") equilibria.

Equilibria occur when $\dot{x} = \ddot{x} = 0$, so $x = 0$. This equilibrium is unstable (it corresponds to a potential energy saddle point, so nearby configurations exist that lead away from equilibrium). The energy is

$$\begin{aligned} 0 &= \ddot{x} + x^2 \\ 0 &= \dot{x}\ddot{x} + \dot{x}x^2 \\ E &= \frac{1}{2}\dot{x}^2 + \frac{1}{3}x^3 \end{aligned}$$

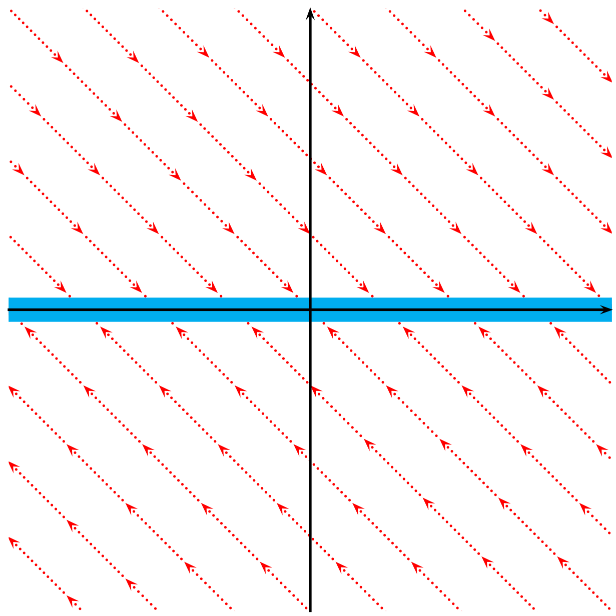
When $x > 0$, the system appears stable, so we would expect trajectories that are "ellipse-like," as was the case for the stable linear spring. When $x < 0$, the system appears unstable, so we would expect trajectories that are "hyperbola-like," as was the case in the previous problem. The equilibrium energy is $E = 0$, which corresponds to the curve $v^2 = \sqrt{-x^3}$.



Problem 4

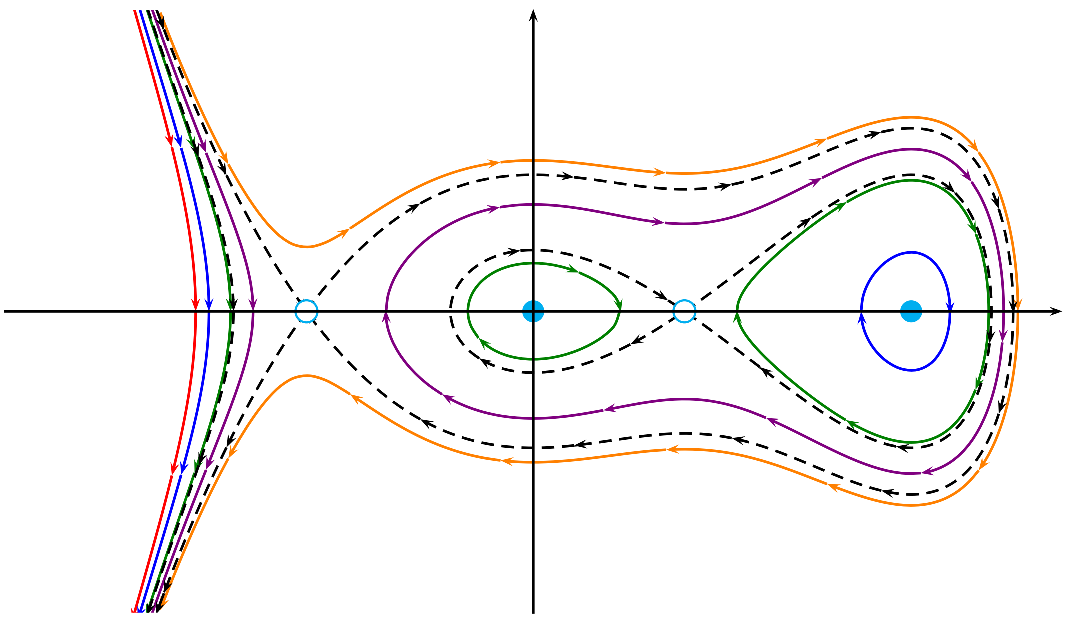
Sketch the phase plane for the ODE $\ddot{x}+\dot{x}=0$. Your sketch should include representative trajectories with arrows, including trajectories through unstable equilibria (if any). Mark all stable ("•") and unstable ("◦") equilibria.

The easiest way to plot the phase plane for this one is to solve it. The equation is $\dot{v}+v=0$, which has the solution $v=v_0e^{-t}$. Then, $\dot{x}=v=v_0e^{-t}$, so $x=x_0+v_0(1-e^{-t})$. Combining these, $x+v=x_0+v_0$, so the trajectories are straight lines with slope -1. Since $v_0=0$ always corresponds to no force ($\ddot{x}=0$ when $v_0=0$), the entire $v=0$ axis is an equilibrium. The solution for v is exponential decay to $v=0$, so the equilibrium is stable.

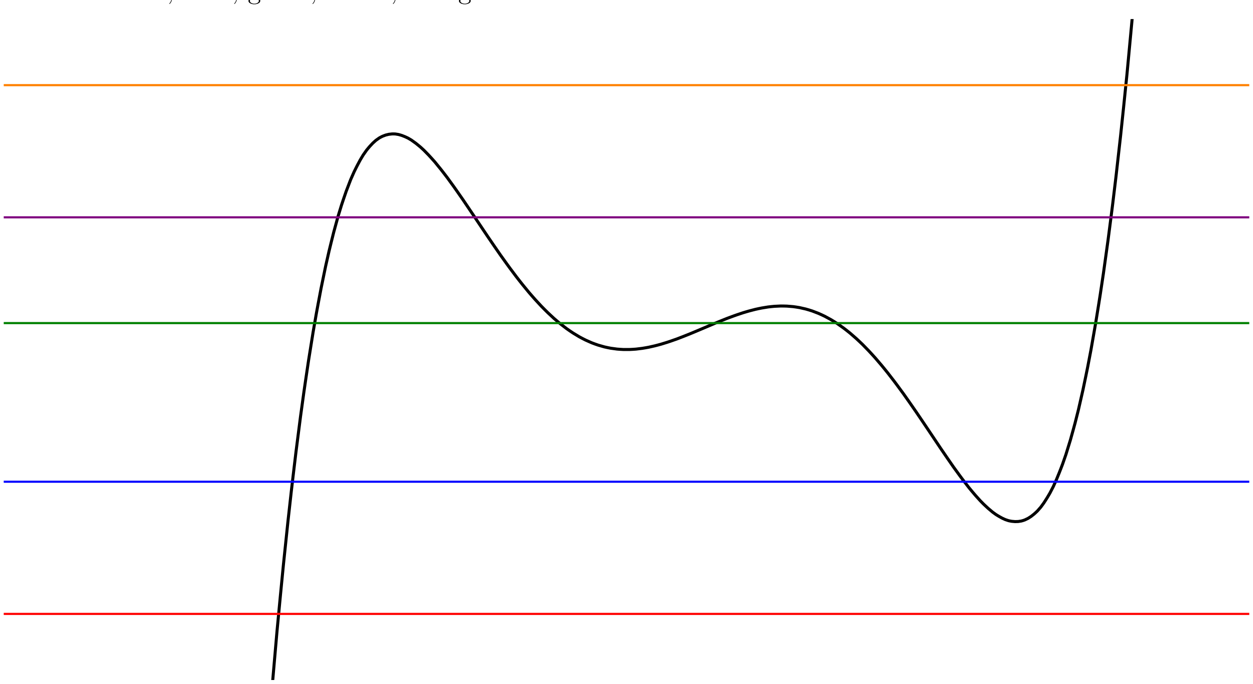


Problem 5

Five energy levels for a system are shown in the phase plane below. (a) List the energy levels (red, orange, green, blue, violet) in order from lowest energy to highest energy. (b) Mark all stable ("•") and unstable ("○") equilibria. (c) Sketch energy contours corresponding to all unstable equilibria (energy contours may contain more than one component; be sure to sketch them all). (d) Add arrows to all contours, including the ones you have added. (e) Sketch the potential energy function and show the energy levels corresponding to the five colored energy contours.



Order: red, blue, green, violet, orange.



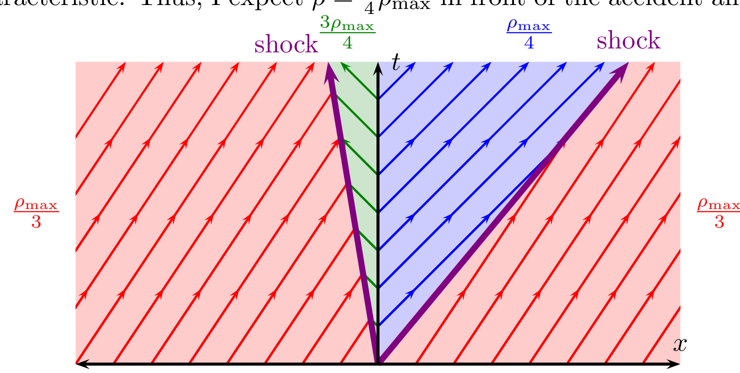
Problem 6

A long road has an initial uniform traffic density $\rho(x,0) = \frac{\rho}{3}$. At $t = 0$, a traffic accident occurs at $x = 0$, which effectively limits the flow rate past $x = 0$ to $q(0,t) = \frac{3}{16} u_{\max} \rho_{\max}$. Determine the traffic density for $t > 0$. Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.

This is essentially the same problem as the light turning red. The only difference is the flow rate. First, we find the densities that correspond to the flux for the accident location.

$$
\begin{aligned}
q &= \rho u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) \\
\rho u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) &= \frac{3}{16} u_{\max} \rho_{\max} \\
\rho^2 - \rho\rho_{\max} + \frac{3}{16}\rho_{\max}^2 &= 0 \\
\left(\rho - \frac{1}{4}\rho_{\max}\right)\left(\rho - \frac{3}{4}\rho_{\max}\right) &= 0 \\
\rho &= \frac{1}{4}\rho_{\max}, \frac{3}{4}\rho_{\max}
\end{aligned}
$$

Of these, $\rho = \frac{1}{4}\rho_{\max}$ corresponds to a forward moving characteristic and $\rho = \frac{3}{4}\rho_{\max}$ corresponds to a backward moving characteristic. Thus, I expect $\rho = \frac{1}{4}\rho_{\max}$ in front of the accident and $\rho = \frac{3}{4}\rho_{\max}$ behind it.



Label the regions ρ_r, ρ_b, and ρ_g (red, green, blue).

$$
\begin{aligned}
\rho_r &= \frac{\rho_{\max}}{3} & q_r &= \rho_r u_{\max}\left(1 - \frac{\rho_r}{\rho_{\max}}\right) = \frac{2}{9}\rho_{\max} u_{\max} \\
\rho_g &= \frac{3\rho_{\max}}{4} & q_g &= \frac{3}{16}\rho_{\max} u_{\max} \\
\rho_b &= \frac{\rho_{\max}}{4} & q_b &= \frac{3}{16}\rho_{\max} u_{\max}
\end{aligned}
$$

Next, we compute the shock speeds for the red-green and blue-red shocks.

$$
\begin{aligned}
\frac{ds_{rg}}{dt} &= \frac{q_r - q_g}{\rho_r - \rho_g} = \frac{\frac{2}{9}\rho_{\max} u_{\max} - \frac{3}{16}\rho_{\max} u_{\max}}{\frac{\rho_{\max}}{3} - \frac{3\rho_{\max}}{4}} = -\frac{1}{12} u_{\max} \\
\frac{ds_{br}}{dt} &= \frac{q_b - q_r}{\rho_b - \rho_r} = \frac{\frac{3}{16}\rho_{\max} u_{\max} - \frac{2}{9}\rho_{\max} u_{\max}}{\frac{\rho_{\max}}{4} - \frac{\rho_{\max}}{3}} = \frac{5}{12} u_{\max}
\end{aligned}
$$

Finally, we can assemble the solution.

$$\rho(x,t) = \begin{cases} \frac{\rho_{\max}}{3} & x < -\frac{1}{12}u_{\max}t \\ \frac{3\rho_{\max}}{4} & -\frac{1}{12}u_{\max}t < x < 0 \\ \frac{\rho_{\max}}{4} & 0 < x < \frac{5}{12}u_{\max}t \\ \frac{\rho_{\max}}{3} & \frac{5}{12}u_{\max}t < x \end{cases}$$

Problem 7

Solve the PDE $\dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial x} + z + t = 0$ subject to $z(x,0) = f(x)$.

Use the method of characteristics. Let $y(t)$ be the path of an observer.

$$\begin{aligned}
\frac{d}{dt}z(y(t),t) &= \frac{\partial z}{\partial t}(y(t),t) + \frac{\partial z}{\partial x}(y(t),t)\frac{dy}{dt}(t) \\
&= \frac{\partial z}{\partial t}(y(t),t) + \frac{\partial z}{\partial x}(y(t),t) \qquad\qquad \text{where } \frac{dy}{dt} = 1 \\
&= -z - t \\
z' &= -z - t \\
z &= Ae^{-t} + rt + s \qquad\qquad \text{(Solve } z' = -z\text{, guess at rest of solution.)} \\
z' &= -Ae^{-t} + r \\
-Ae^{-t} + r &= -(Ae^{-t} + rt + s) - t \\
r + s &= -rt - t \\
r &= -1 \\
s &= 1 \\
z &= Ae^{-t} - t + 1 \\
z_0 &= z(0) = A + 1 \\
z &= (z_0 - 1)e^{-t} - t + 1
\end{aligned}$$

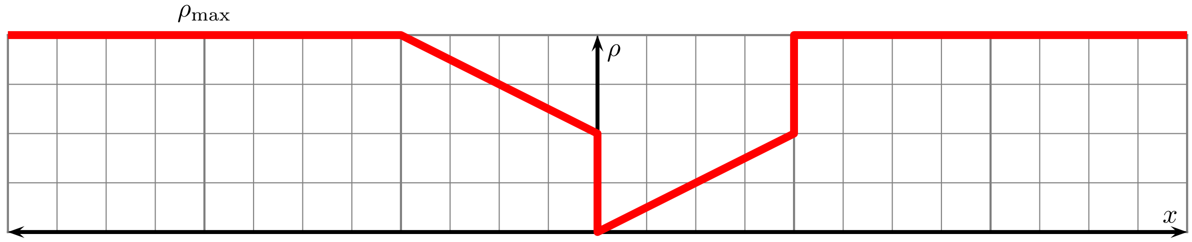
Solving the ODE for $y(t)$ we have $y(t) = t + y_0$.

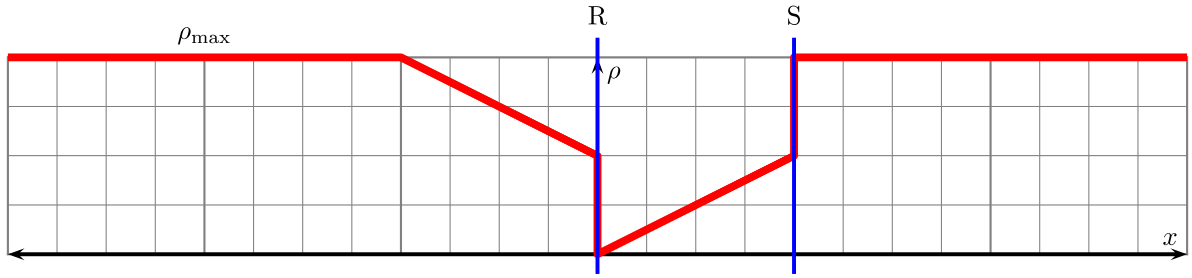
Now, we can compute $z(x,t)$. First, we must compute y_0, the observer's starting location. $x = y(t) = t + y_0$ implies $y_0 = x - t$. Then, $z_0 = z(y_0, 0) = f(x - t)$. Finally,

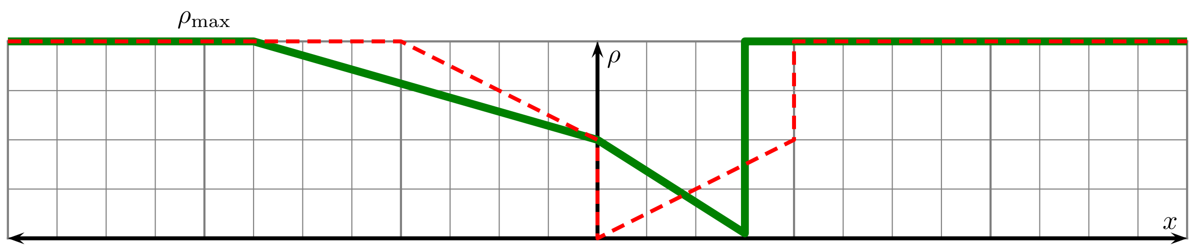
$$z(x,t) = (f(x-t) - 1)e^{-t} - t + 1$$

Problem 8

Identify the location of the shock ("S") and rarefaction ("R") in the initial density profile (red line). Sketch the density profile at the time when the shock and rarefaction first meet. Try to be accurate in your sketch, but do not attempt to solve the PDE analytically. Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.

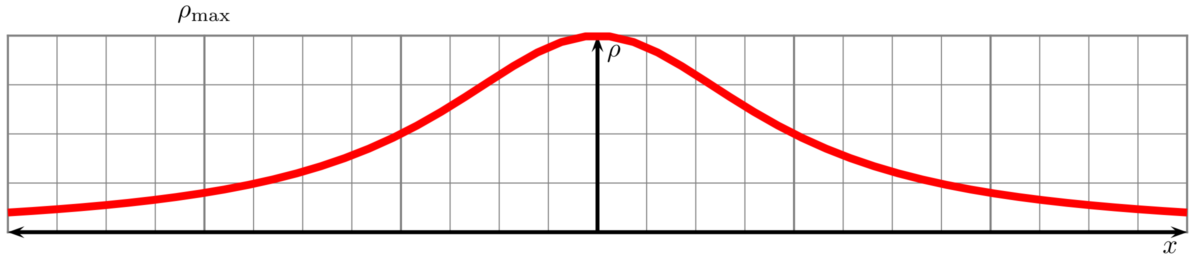


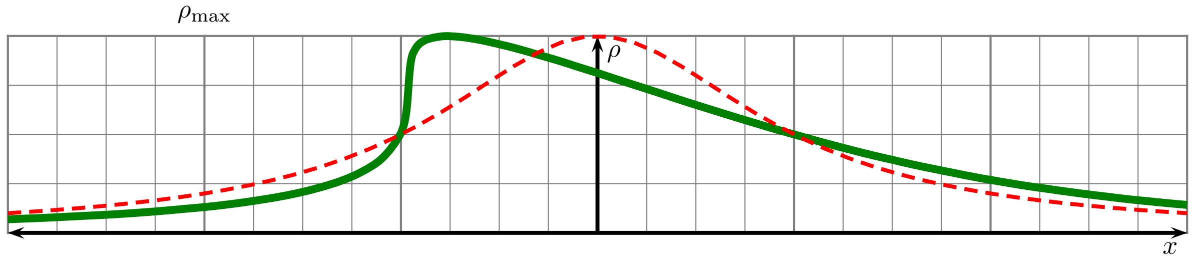




Problem 9

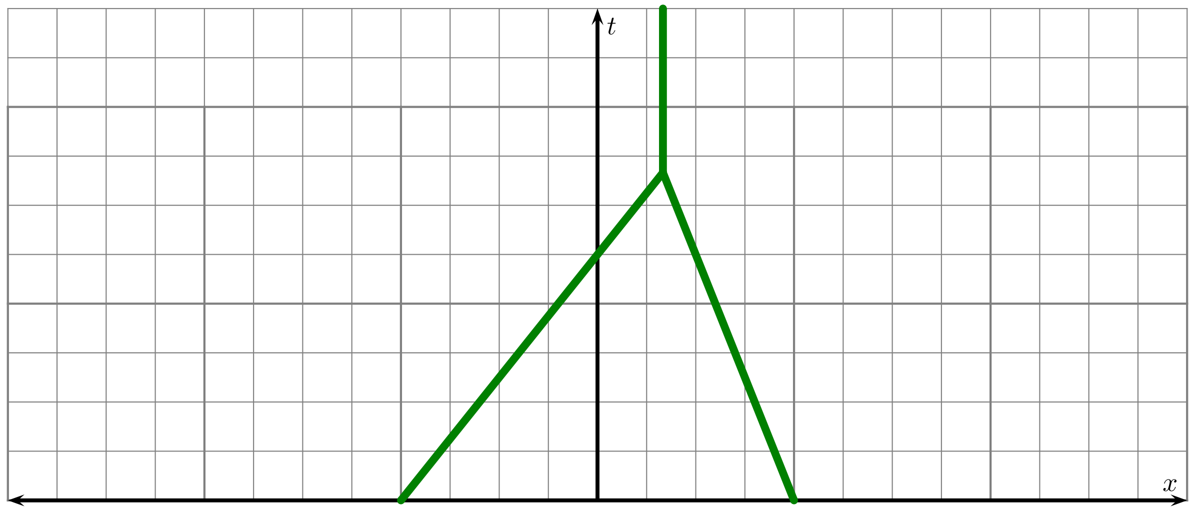
Sketch the density distribution at the time of the first shock. Try to be accurate in your sketch, but do not attempt to solve the PDE analytically. Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.





Problem 10

The green lines in the illustration below show the locations of the shocks that occur when some piecewise constant initial density profile evolves in time. There are no rarefactions. Sketch the initial density profile and indicate which portions represent light or heavy traffic. Be sure to explain your reasoning. Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.



Since the initial density is piecewise constant, each change in density will be a discontinuity. A discontinuity where density decreases is a rarefaction, which does not occur here. Discontinuities where density increases are shocks, of which there are initially two. Thus, the initial density profile has three parts, which meet at the initial shock locations. The profile is increasing. Lets call them $\rho_0 < \rho_1 < \rho_2$.

The tools needed to estimate the traffic's density are that (a) light traffic has characteristics that move forward while the opposite happens for heavy traffic and (b) for the car following model $\hat{u}(\rho)$ we are using, the shock velocity is the average of the characteristic velocities for the densities that it separates. Let the characteristics be $c_0 > c_1 > c_2$.

Since the merged shock does not move, $\frac{c_0+c_2}{2} = 0$, which immediately implies $c_0 > 0 > c_2$. Let the velocity of the left shock be s_0 and the velocity of the right shock be s_2. It is clear from the diagram that $|s_0| > |s_2|$ and $s_0 > 0 > s_1$, so $s_0 + s_1 > 0$. Then, $\frac{c_0+c_1}{2} + \frac{c_1+c_2}{2} > 0$, which leads to $c_1 > 0$. ρ_0 and ρ_1 represent light traffic. ρ_2 represents heavy traffic. Note that $\rho_0 + \rho_2 = \rho_{\max}$.