Math 142-2, Project 2

Your name here

Programming

In this project, you will implement the Lax-Friedrichs numerical scheme

$$z_i^{n+1} = \frac{1}{2}(z_{i+1}^n + z_{i-1}^n) - \frac{\Delta t}{2\Delta x}(f(z_{i+1}^n) - f(z_{i-1}^n))$$

for numerically solving the PDE

$$\begin{aligned} \frac{\partial z}{\partial t} + \frac{\partial}{\partial x}(f(z)) &= 0 \\ f(z(x_l)) &= F_l \\ f(z(x_r)) &= F_r \end{aligned}$$

for $x \in [x_l, x_r]$ and for $t \in [0, T]$. This will be quite helpful to you later on in the course, when you will be able to use this numerical method to check your answers on homework problems. Using this tool to get some intuition for how the PDEs we discuss in this course evolve may also be helpful to you on homework and exams.

Your numerical scheme will require careful treatment of the boundaries. If you have valid data for z_m^n for $1 \leq m \leq M-1$, then the numerical scheme above will require z_0^n and z_M^n to compute z_m^{n+1}. Note that flux boundary conditions enter into the numerical method through $f(z_0^n) = F_l$ and $(z_M^n) = F_r$. But this scheme still requires z_0^n and z_M^n. For this, we want to use the approximations $z_0^n = z_1^n$ and $z_M^n = z_{M-1}^n$.

Problem 1

Show that the scheme above can be written as

$$\frac{z_i^{n+1} - \frac{1}{2}(z_{i+1}^n + z_{i-1}^n)}{\Delta t} + \frac{f(z_{i+1}^n) - f(z_{i-1}^n)}{2\Delta x} = 0$$

Your solution goes here

Problem 2

Show that, if z is smooth, then the PDE we are solving can be written in the form

$$\frac{\partial}{\partial t}\int_a^b z\,dx + f(z(b,t)) - f(z(a,t)) = 0.$$

This integral form of the PDE is called the *weak form* because its requirements on z are weaker; no spatial derivatives of z are required, so z need not be differentiable in space. In fact, z need not even be continuous. You will need to use this form to show that non-smooth solutions satisfy the PDE. Such non-smooth solutions are called *weak solutions* (solutions to the original differential form of the PDE are called *strong solutions*). Weak solutions must satisfy the above equation for any choices a and b.

Your solution goes here

Problem 3

Consider the PDE

$$\frac{\partial z}{\partial t} + c\frac{\partial z}{\partial x} = 0 \qquad z(x,0) = z_0(x)$$

where c is a constant and $x \in (-\infty, \infty)$. Show that $z(x,t) = z_0(x - ct)$ is a *weak* solution to this PDE, even if z_0 (and thus z) is not differentiable.

Your solution goes here

Problem 4

In the PDE from the previous problem, let $c \in \{1, 2, 3, -1\}$. Use the domain $x \in [-1, 1]$ and $t \in [0, \frac{1}{4}]$, choosing boundary conditions (F_l and F_r) so that $z(x,t) = |x - ct|$ is the analytic solution. Take time steps $\Delta t = k\Delta x$, for $\Delta x \in \{2^{-6}, 2^{-8}, 2^{-10}\}$ while fixing k. What happens numerically when $k|c| > 1$? What happens numerically when $k|c| < 1$? For each c, plot the analytic solution along with the solution for each choice of Δx using $k = 0.9/|c|$. Repeat the process with $k = 1.1/|c|$.

Your solution goes here

Problem 5

The numerical scheme above does not work for all choices of Δt and Δx. The following restriction must be enforced

$$\Delta t < \frac{\Delta x}{\max\limits_{x} |f'(z(x,t))|}$$

For each of the PDEs (defined by the flux function $f(z)$) and analytic solutions in the table below plot the numerical solution that you obtain superimposed with the analytic solution. If your numerical scheme is implemented correctly and Δx is large enough, the two should be very close. In each case, $x \in [-1, 1]$ and $t \in [0, \frac{1}{2}]$.

#	$f(z)$	$z(x,t)$
1	z	$\|x-t\|$
2	$-z$	$\|x+t\|$
3	$2z$	$\|x-2t\|$
4	$\frac{1}{2}z^2$	1
5	$\frac{1}{2}z^2$	$\frac{x}{t+1}$
6	$\frac{1}{2}z^2$	$\frac{x}{t-2}$
7	$\frac{1}{2}z^2$	$\begin{cases} 1 & x<0 \\ -1 & x>0 \end{cases}$
8	$\frac{1}{2}z^2$	$\begin{cases} -1 & x<-t \\ \frac{x}{t} & -t<x<t \\ 1 & x \geq t \end{cases}$
9	$\frac{1}{2}z^2$	$\begin{cases} 2 & x<t \\ 0 & x>t \end{cases}$

Your solution goes here

Problem 6

Show that $z(x,t) = \begin{cases} 2 & x<0 \\ 0 & x>0 \end{cases}$ is *not* a weak solution to the PDE $z_t + zz_x = 0$.

Your solution goes here

Problem 7

Show that $z(x,t) = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$ is a weak solution to the PDE $z_t + zz_x = 0$. Your numerical implementation does not produce this solution; the solution that the numerical method will produce is in the table above. Since there are at least two solutions, this PDE need not have unique weak solutions.

Your solution goes here

Final Notes

At this point, you should have a working numerical method for these sorts of PDEs, which you will be using for the third project. You may also use it to verify homework problems and generally understand how the solution to PDEs will behave.