Math 142-2, Homework 6

Your name here

Problem 71.1

Experiments in the Lincoln Tunnel (combined with the theoretical work discussed in exercise 63.7) suggest that the traffic flow is approximately

$$q(\rho) = a\rho[\ln(\rho_{\max}) - \ln(\rho)]$$

(where a and $\rho_{\max}$ are known constants). Suppose the initial density $\rho(x, 0)$ varies linearly from bumper-to-bumper traffic (behind $x = -x_0$) to no traffic (ahead of $x = 0$) as sketched in Fig. 71-6. Two hours later, where does $\rho = \rho_{\max}/2$?

Your solution goes here

Problem 71.9

Show that $\rho = f(x - q'(\rho)t)$ satisfies equation 71.1 for any function f. Note that initially $\rho = f(x)$. Briefly explain how this solution was obtained.

Your solution goes here

Problem 72.5

Sketch $dq/d\rho$ as a function of x, for fixed $t > 0$ (after the light turns green).

Your solution goes here

Problem 73.1

Assume that the traffic density is initially

$$\rho(x, 0) = \begin{cases} \rho_{\max} & x < 0 \\ \rho_{\max}/2 & 0 < x < a \\ 0 & a < x. \end{cases}$$

Sketch the initial density. Determine and sketch the density at all later times. Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$.

Your solution goes here

Problem 73.6

Assume

$$u = u_{\max}\left(1 - \frac{\rho^2}{\rho_{\max}^2}\right).$$

Determine the traffic density that results after an infinite line of stopped traffic is started by a red traffic light turning green.

Your solution goes here

Problem 73.9

At what velocity does the information that the traffic light changed from red to green travel?

Your solution goes here

Problem 74.2

Assume $u(\rho) = u_{\max}(1 - \rho^2/\rho_{\max}^2)$ and

$$\rho(x,0) = \begin{cases} \rho_{\max} & x < 0 \\ \rho_{\max}(L-x)/L & 0 < x < L \\ 0 & L < x. \end{cases}$$

Determine $\rho(x,t)$.

Your solution goes here

Problem 74.3

Consider the following partial differential equation:

$$\frac{\partial \rho}{\partial t} - \rho^2 \frac{\partial \rho}{\partial x} = 0 \qquad -\infty < x < \infty.$$

(a) Why can't this equation model a traffic flow problem?

Your solution goes here

(b) Solve this partial differential equation by the method of characteristics, subject to the initial conditions:

$$\rho(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 < x < 1 \\ 0 & 1 < x. \end{cases}$$

Your solution goes here

Problem 74.4

Consider the example solved in this section. What traffic density should be approached as $L \to 0$? Verify that as $L \to 0$ equation 74.9 approaches the correct traffic density.

Your solution goes here

Problem 77.1

If $u = u_{\max}(1 - \rho/\rho_{\max})$, then what is the velocity of a traffic shock separating densities ρ_0 and ρ_1? (Simplify the expression as much as possible.) Show that the shock velocity is the average of the density wave velocities associated with ρ_0 and ρ_1.

Your solution goes here

Problem 77.2

If $u = u_{\max}(1 - \rho^2/\rho_{\max}^2)$, then what is the velocity of a traffic shock separating densities ρ_0 and ρ_1? (Simplify the expression as much as possible.) Show that the shock velocity is not the average of the density wave velocities associated with ρ_0 and ρ_1.

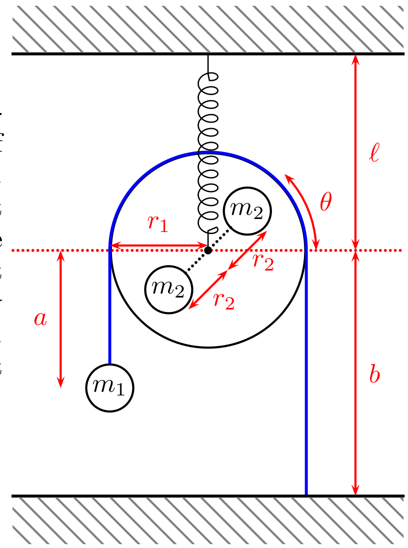
Your solution goes here

Problem 77.3

A weak shock is a shock in which the shock strength (the difference in densities) is small. For a weak shock, show that the shock velocity is approximately the average of the density wave velocities associated with the two densities. [Hint: Use Taylor series methods.]

Your solution goes here

Additional Problem



A pulley of radius r_1 hangs from a spring (rest length ℓ, spring constant k) from the ceiling of a room as shown at right. The center of the pulley is at $(0, y(t))$, and the top of the spring is fixed at $(0, \ell)$. The pulley is able to rotate around the end of the spring without friction. A cable extends around the pulley with a weight m_1 at one end (located at $(-r_1, z(t))$) and the other end fixed to the floor at $(r_1, -b)$. Two identical point masses m_2 are attached to the pulley at a distance r_2 from the pulley's center at angles $\theta(t)$ and $\theta(t) + \pi$. Assume the cable has negligible mass. When the spring is at its rest length, $y(t) = 0$, $z(t) = -a$, and $\theta(t) = \frac{\pi}{4}$. You may assume that m_1 never touches the pulley or the ground.

(a) Express $z(t)$ and $\theta(t)$ in terms of $y(t)$.

Your solution goes here

(b) Express the potential energy $\phi_1(t)$ and kinetic energy $K_1(t)$ of the mass m_1 in terms of $y(t)$ and $y'(t)$.

Your solution goes here

(c) Express the potential energy $\phi_2(t)$ and kinetic energy $K_2(t)$ of the two masses m_2 in terms of $y(t)$ and $y'(t)$.

Your solution goes here

(d) Express the potential energy $\phi_3(t)$ of the spring in terms of $y(t)$.

Your solution goes here

(e) The total energy is $E = \phi_1 + \phi_2 + \phi_3 + K_1 + K_2$. Locate the equilibrium y_{eq}.

Your solution goes here

(f) Let $w(t) = y(t) - y_{eq}$, where y_{eq} is the equilibrium configuration. Show that $w(t)$ is described by the same ODE $\hat{m}w'' + \hat{c}w' + \hat{k}w = 0$ that was used to describe a simple spring. What are the constants $\hat{m}$, $\hat{c}$, and $\hat{k}$?

Your solution goes here