Math 142-2, Homework 5

Your name here

Problem 63.6

Assume that $\hat{u} = \hat{u}(\rho)$. If α equals a car's acceleration, show that

$$\alpha = -\rho \frac{d\hat{u}}{d\rho} \frac{\partial u}{\partial x}.$$

Is the minus sign reasonable? Note that I have given $\hat{u}$ and u different names to make their different functional dependence explicit. In particular, $u(x(t), t) = \hat{u}(\rho(x(t), t))$.

Your solution goes here

Problem 63.7

Consider exercise 61.3. Suppose that drivers accelerate such that

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{a^2}{\rho} \frac{\partial \rho}{\partial x},$$

where a is a positive constant.

(a) Physically interpret this equation.

Your solution goes here

(b) If u only depends on ρ and the equation of conservation of cars is valid, show that

$$\frac{du}{d\rho} = -\frac{a}{\rho}.$$

Note: You may assume $\frac{d\hat{u}}{d\rho} < 0$.

Your solution goes here

(c) Solve the differential equation in part (b), subject to the condition that $u(\rho_{\max}) = 0$. The resulting flow-density curve fits quite well to the Lincoln Tunnel data.

Your solution goes here

(d) Show that a is the velocity which corresponds to the road's capacity.

Your solution goes here

(e) Discuss objections to this theory for small densities.

Your solution goes here

Problem 64.1

Consider the linear car-following model, equation 64.2, with a response time T (a delay).

(a) Ignore this part.

(b) Assume the lead driver's velocity varies periodically

$$v_0 = \mathrm{Re}(1 + f_0 e^{i\omega t}).$$

Also assume the n-th driver's velocity varies periodically

$$v_n = \mathrm{Re}(1 + f_n e^{i\omega t}),$$

where f_n measures the amplification or decay which occurs. Show that

$$f_n = \left(1 + \frac{i\omega}{\lambda} e^{i\omega T}\right)^{-n} f_0,$$

where $0 < f_0 < 1$. Note: f_n will be complex, f_0 is real. Unlike the problem as stated in the book, I am assuming $f_0 < 1$, since otherwise the model predicts that even the first car stops frequently. If $f_0 \approx 0$, then the first car drives quite smoothly, and the effects on later cars are more meaningful. This alteration does not make the problem more difficult.

Your solution goes here

(c) Show the magnitude of the amplification factor f_n decreases with n if

$$\frac{\sin \omega T}{\omega} < \frac{1}{2\lambda}.$$

Your solution goes here

(d) Show that the above inequality holds for all ω only if $\lambda T < \frac{1}{2}$.

Your solution goes here

(e) Conclude that if the product of the sensitivity and the time lag is greater than $\frac{1}{2}$, it is possible for following cars to drive much more erratically than the leader. In this case we say the model predicts instability if $\lambda T > \frac{1}{2}$ (i.e., with a sufficiently long time lag). (This conclusion can be reached more expeditiously through the use of Laplace transforms.)

Your solution goes here

Problem 65.1

Determine the solution of $\frac{\partial \rho}{\partial t} = (\sin x)\rho$ which satisfies $\rho(x, 0) = \cos x$.

Your solution goes here

Problem 67.3

Suppose initially ($t = 0$) that the traffic density is $\rho = \rho_0 + \epsilon \sin x$, where $|\epsilon| \ll \rho_0$. Determine $\rho(x, t)$.

Your solution goes here

Problem 67.5

Based on a linear analysis, would you say $\rho = \rho_0$, a constant, is a stable or unstable equilibrium solution of equation 66.1?

Your solution goes here

Problem 68.1

Explain why a density wave moves forward for light traffic. Consider both cases in which the traffic is getting heavier down the road and lighter.

Your solution goes here