Math 142-2, Final

Your name: ________________________

Student ID: ________________________

Instructions

This exam is closed book. No notes, books, electronic devices, or other resources are permitted on this exam. Be sure to write your name and student ID number at the top of this page. Scratch paper will provided for you to work out problems and write your answers. When you finish the exam, please staple all of the scratch paper that you have written on (even if it does not contain answers) to this question sheet when you turn in your exam.

Problem 1

A fictitious organism happens to come in three types (similar to how many animals come in male and female), which we can label A, B, and C. These organisms have very peculiar reproduction habits.

- **Type A and type B can together reproduce to form type C.**
- **Type B and type C can together reproduce to form type A.**
- **Type C and type A can together reproduce to form type B.**
- **Two members of the same type reproduce to form offspring of the same type.**
- **These organisms are unaware that they come in three types, as all members look and behave alike.**
- **The organisms adjust their birth rates so as to maintain a constant total population T.**

All types have the same death rates. Devise a model that describes the populations $A(t)$, $B(t)$, and $C(t)$ and is consistent with the above observations. ***(5 points)***

Let r be a factor by which the overall reproduction is reduced to maintain the population. Let s be the death rate.

$$\begin{aligned}
\frac{dA}{dt} &= 2rBC + rA^2 - sA \\
\frac{dB}{dt} &= 2rCA + rB^2 - sB \\
\frac{dC}{dt} &= 2rAB + rC^2 - sC \\
0 &= \frac{dA}{dt} + \frac{dB}{dt} + \frac{dC}{dt} \\
&= r(A^2 + B^2 + C^2 + 2AB + 2CA + 2BC) - s(A + B + C) \\
&= r(A + B + C)^2 - s(A + B + C) \\
&= rT^2 - sT \\
r &= \frac{s}{T} \\
r &= \frac{s}{A + B + C} \\
\frac{dA}{dt} &= s\left(\frac{2BC + A^2}{A + B + C} - A\right) \\
\frac{dB}{dt} &= s\left(\frac{2CA + B^2}{A + B + C} - B\right) \\
\frac{dC}{dt} &= s\left(\frac{2AB + C^2}{A + B + C} - C\right)
\end{aligned}$$

Problem 2

Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$**.**

(a) Show that if $\rho(x, 0) = \rho_{\max} - \rho(a - x, 0)$ **then** $\rho(x, t) = \rho_{\max} - \rho(a - x, t)$ **for all** $t \geq 0$**. *(5 points)***

Let $\hat{\rho}(x,t) = \rho_{\max} - \rho(a-x,t)$ and $y = a - x$.

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\rho}(x,t) &= \frac{\partial}{\partial t}(\rho_{\max} - \rho(a-x,t)) \\
&= -\frac{\partial}{\partial t}\rho(a-x,t) \\
&= -\frac{\partial \rho}{\partial t}(y,t) \\
\frac{\partial}{\partial x}\hat{\rho}(x,t) &= \frac{\partial}{\partial x}(\rho_{\max} - \rho(a-x,t)) \\
&= -\frac{\partial}{\partial x}\rho(a-x,t) \\
&= \frac{\partial \rho}{\partial x}(y,t) \\
\frac{dq}{d\rho}(\hat{\rho}(x,t)) &= u_{\max}\left(1 - \frac{2\hat{\rho}(x,t)}{\rho_{\max}}\right) \\
&= u_{\max}\left(1 - \frac{2(\rho_{\max} - \rho(a-x,t))}{\rho_{\max}}\right) \\
&= -u_{\max}\left(1 - \frac{\rho(a-x,t)}{\rho_{\max}}\right) \\
&= -\frac{dq}{d\rho}(\hat{\rho}(y,t)) \\
\frac{\partial \hat{\rho}}{\partial t} + \frac{dq}{d\rho}(\hat{\rho})\frac{\partial \hat{\rho}}{\partial x} &= -\frac{\partial \rho}{\partial t}(y,t) - \frac{dq}{d\rho}(\rho(y,t))\frac{\partial \rho}{\partial x}(y,t) \\
&= -\left(\frac{\partial \rho}{\partial t}(y,t) + \frac{dq}{d\rho}(\rho(y,t))\frac{\partial \rho}{\partial x}(y,t)\right) \\
&= 0
\end{aligned}$$

Thus, if $\rho(x,t)$ is a solution to the differential equation with initial data $\rho(x,0)$, then $\hat{\rho}(x,t) = \rho_{\max} - \rho(a-x,t)$ is a solution to the same differential equation with initial data $\hat{\rho}(x,0)$. Since $\rho(x,0) = \hat{\rho}(x,0)$, the solutions must be the same. Thus $\rho(x,t) = \rho_{\max} - \rho(a-x,t)$ for all $t \geq 0$.

(b) Let $\rho(x,0) = \frac{\rho_{\max}}{2}(1+\sin(x))$ for the remaining parts of this problem. At $t = 0$, where will the cars move slowest? ***(5 points)***

The car velocity u is minimized at $\rho = \rho_{\max}$, where they are stopped. This occurs at $x = (2n + \frac{1}{2})\pi$.

(c) At $t = 0$, where do the traffic waves move slowest? ***(5 points)***

Traffic waves move slowest when $0 = q'(\rho) = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$, or $\rho = \frac{\rho_{\max}}{2}$. This occurs at $x = n\pi$.

(d) When will the first shocks form? ***(5 points)***

The first shock forms at

$$T = \min_x \frac{\rho_{\max}}{2u_{\max}\frac{\partial\rho}{\partial x}(x,0)} = \frac{\rho_{\max}}{2u_{\max}\max_x \frac{\partial\rho}{\partial x}(x,0)} = \frac{1}{u_{\max}}.$$

(e) Where will the first shocks form? ***(5 points)***

The first shocks will form from the characteristics that start where $\frac{\partial\rho}{\partial x}(x,0)$ is maximized, which occurs at $x_0 = 2n\pi$. Since $\rho_0 = \rho(x_0,0) = \frac{\rho_{\max}}{2}$, these characteristics move with velocity $q'(\rho_0) = 0$. Since the characteristics corresponding to the first shocks do not move, the shocks will form at $x = 2n\pi$. Note that the answer to (d) is not needed to answer this question.

(f) With what velocity will those shocks move? ***(5 points)***

The shocks are stationary. The symmetry from (a) ensures this.

An alternative strategy is to note that the characteristics from x_0 and $-x_0$ reach the origin at the same time and represent different densities but the same flux.

$$\begin{aligned}
\frac{dq}{d\rho} &= u_{\max}\left(1 - \frac{2\rho}{\rho_m ax}\right) \\
\frac{dq}{d\rho}(\rho(x_0,0)) &= u_{\max}\left(1 - \frac{2\frac{\rho_{\max}}{2}(1+\sin(x_0))}{\rho_m ax}\right) \\
&= u_{\max}(1-(1+\sin(x_0))) \\
&= -u_{\max}\sin(x_0) \\
z(t) &= -u_{\max}\sin(x_0)t + x_0
\end{aligned}$$

We are interested in $z(t)=0$, and we note that if $z(t)=0$ for some x_0 and t, then would also be zero with $-x_0$ and t. These are the two characteristics that will meet at the shock that formed at $x=0$.

$$\begin{aligned}
q &= u_{\max}\rho\left(1 - \frac{\rho}{\rho_m ax}\right) \\
q(\rho(x_0,0)) &= u_{\max}\frac{\rho_{\max}}{2}(1+\sin(x_0))\left(1 - \frac{\frac{\rho_{\max}}{2}(1+\sin(x_0))}{\rho_m ax}\right) \\
&= \frac{\rho_{\max}u_{\max}}{2}(1+\sin(x_0))\left(1 - \frac{1}{2}(1+\sin(x_0))\right) \\
&= \frac{\rho_{\max}u_{\max}}{4}(1+\sin(x_0))(2-(1+\sin(x_0))) \\
&= \frac{\rho_{\max}u_{\max}}{4}(1+\sin(x_0))(1-\sin(x_0))
\end{aligned}$$

Then, we see that $q(\rho(x_0,0)) = q(\rho(-x_0,0))$ but $\rho(x_0,0) \neq \rho(-x_0,0)$. Finally

$$\frac{ds}{dt} = \frac{q_+ - q_-}{\rho_+ - \rho_-} = \frac{q(\rho(x_0,0)) - q(\rho(-x_0,0))}{\rho(x_0,0) - \rho(-x_0,0)} = 0.$$

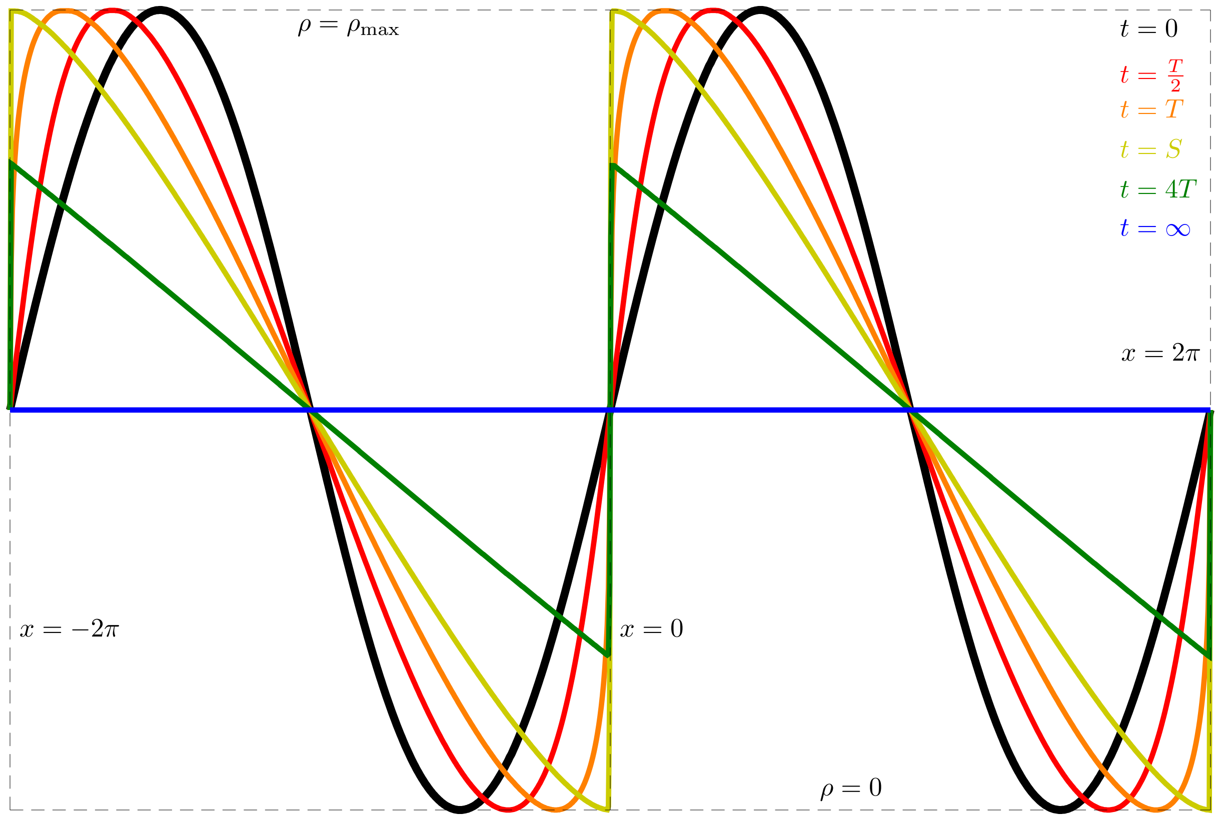
Note that if you guessed that (e) was 0, for example by a symmetry argument, then $0 = -u_{\max}\sin(x_0)t + x_0$ yields the answer to (d) directly:

$$\begin{aligned}
0 &= -u_{\max}\sin(x_0)t + x_0 \\
u_{\max}\sin(x_0)t &= x_0 \\
t &= \frac{x_0}{u_{\max}\sin(x_0)} \\
0 &= \frac{dt}{dx_0} \\
&= \frac{\frac{dx_0}{dx_0}(u_{\max}\sin(x_0)) - x_0\frac{d}{dx_0}(u_{\max}\sin(x_0))}{(u_{\max}\sin(x_0))^2} \\
&= \frac{u_{\max}\sin(x_0) - x_0 u_{\max}\cos(x_0)}{(u_{\max}\sin(x_0))^2} \\
0 &= \sin(x_0) - x_0\cos(x_0) \\
x_0 &= \tan(x_0) \\
x_0 &= 0 \\
t &= \lim_{x_0\to 0}\frac{x_0}{u_{\max}\sin(x_0)} \\
\frac{1}{t} &= u_{\max}\lim_{x_0\to 0}\frac{\sin(x_0)}{x_0} \\
&= u_{\max} \\
t &= \frac{1}{u_{\max}}
\end{aligned}$$

(g) For how long S will there be stopped cars? *(5 points)*

Consider the stopped cars at $x = \frac{\pi}{2}$ (the situation is periodic, so all of places with stopped cars will behave the same). The characteristic velocity here is $-u_{\max}$, so the stopped cars will be at $x = \frac{\pi}{2} - u_{\max}t$ until these cars reach the shock at $x = 0$. This occurs at time $t = \frac{\pi}{2u_{\max}}$. At this point, these characteristics are swallowed by the shock, and there are no longer stopped cars.

(h) If T is the time at which the first shock forms, sketch the traffic density for $-2\pi \leq x \leq 2\pi$ at $t = 0$, $t = \frac{T}{2}$, $t = T$, $t = S$, $t = 4T$, and $t \to \infty$. Do not attempt to solve the PDE analytically. *(5 points)*



Problem 3

The PDE $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$ is called the inviscid Burgers' equation. Because of its very simple and relatively intuitive behavior, it is classically studied as a way of understanding how PDEs behave, studying shocks and rarefactions, and testing numerical methods.

(a) Write this PDE in conservation form $U_t + F_x = 0$. What is the flux? What is the conserved quantity? *(5 points)*

Conservation form is $U_t + F_x = 0$, so $\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) = 0$ is in conservation form. The flux is $F = \frac{v^2}{2}$, and the conserved quantity is $\int_{-\infty}^{\infty} v\, dx$.

(b) What is the local characteristic velocity? *(5 points)*

The local characteristic velocity is v. (Compare to the PDE for conservation of cars.) The simplicity of the characteristics is what makes it relatively easy to work with.

(c) If $v = \rho$, this PDE can be thought of as the equation for conservation of cars with some traffic following model $u(\rho)$. What is this traffic following model $u(\rho)$? Is this model reasonable? *(5 points)*

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} &= 0 \\
\frac{\partial \rho}{\partial t} + q'(\rho) \frac{\partial \rho}{\partial x} &= 0 \\
q'(\rho) &= \rho \\
q(\rho) &= \frac{1}{2}\rho^2 + c_1 \\
u(\rho) &= \frac{\rho}{2} + \frac{c_1}{\rho}
\end{aligned}$$

If $c_1 \neq 0$, then $u(0)$ is unbounded. On the other hand, if $c_1 = 0$, then $u'(\rho) \geq 0$, which is also objectionable. This model is not reasonable.

(d) Show that the more general transformation $v = a\rho + b$ allows this PDE to be transformed into the equation for conservation of cars with the traffic following model $u = u_{\max}(1 - \rho/\rho_{\max})$; find a and b. In other words, conservation of cars with this simple car following model behaves exactly the same as Burgers' equation; understanding how one evolves immediately provides insight into the behavior of the other. ***(5 points)***

$$\begin{aligned}
0 &= \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \\
&= a\frac{\partial \rho}{\partial t} + a(a\rho + b)\frac{\partial \rho}{\partial x} \\
0 &= \frac{\partial \rho}{\partial t} + (a\rho + b)\frac{\partial \rho}{\partial x} \\
0 &= \frac{\partial \rho}{\partial t} + u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)\frac{\partial \rho}{\partial x} \\
a &= -\frac{2u_{\max}}{\rho_{\max}} \\
b &= u_{\max}
\end{aligned}$$

Problem 4

For each system below, determine the equilibrium points and classify each as stable, neutrally stable, or unstable. ***(5 points each)***

(a) $\dfrac{dx}{dt} = -2x$

The system is linear with $A = \begin{pmatrix} -2 \end{pmatrix}$, so that $\lambda = -2$. The system has an equilibrium at the origin that is stable.

(b) $\dfrac{dx}{dt} = -2y \qquad \dfrac{dy}{dt} = 2x$

The system is linear with $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$. $0 = \begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + 4$, so that $\lambda = \pm 2i$. The system has an equilibrium at the origin that is neutrally stable.

(c) $\dfrac{dx}{dt} = -2z \qquad \dfrac{dy}{dt} = 2x \qquad \dfrac{dz}{dt} = 2y$

The system is linear with

$$\begin{aligned} A &= \begin{pmatrix} 0 & 0 & -2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \\ 0 &= \begin{vmatrix} -\lambda & 0 & -2 \\ 2 & -\lambda & 0 \\ 0 & 2 & -\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} -\lambda & 0 \\ 2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & -2 \\ 2 & -\lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & -2 \\ -\lambda & 0 \end{vmatrix} \\ &= -\lambda^3 - 8 \\ &= -(\lambda + 2)(\lambda^2 - 2\lambda + 4) \\ \lambda &\in \{-2, 1 \pm i\sqrt{3}\} \end{aligned}$$

Since the complex eigenvalues have positive real part, the system has an equilibrium at the origin that is unstable.

(d) $\dfrac{dx}{dt} = e^{xy} \qquad \dfrac{dy}{dt} = x^2 + y^2 - 1$

$e^{xy} = 0$ is not possible, so there are no equilibria for this system.

(e) $\dfrac{dx}{dt} = e^{xy} - 1 \qquad \dfrac{dy}{dt} = x^2 - y^2 - 1$

From $e^{xy} - 1 = 0$, we must have $x = 0$ or $y = 0$. $x^2 - y^2 - 1 = 0$ has no solutions when $x = 0$. When $y = 0$, $x = \pm 1$. There are two equilibria to consider. Next, lets linearize the system about the point $x = x_0$, $y = 0$.

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} &= \begin{pmatrix} \frac{\partial}{\partial x}(e^{xy} - 1) & \frac{\partial}{\partial y}(e^{xy} - 1) \\ \frac{\partial}{\partial x}(x^2 - y^2 - 1) & \frac{\partial}{\partial y}(x^2 - y^2 - 1) \end{pmatrix}_{x = x_0, y = 0} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\ &= \begin{pmatrix} ye^{xy} & xe^{xy} \\ 2x & 2y \end{pmatrix}_{x = x_0, y = 0} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_0 \\ 2x_0 & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \end{aligned}$$

The characteristic equation is $\lambda^2 - 2x_0^2 = 0$, so that $\lambda = \pm x_0\sqrt{2}$. Since $x_0 = \pm 1$, $\lambda = \pm\sqrt{2}$ for both equilibria, making them both unstable.

Problem 5

Consider the model (where $x(t)$ and $y(t)$ are any real numbers)

$$\frac{dx}{dt} = xy \qquad \frac{dy}{dt} = y^2 - 1$$

(a) Locate the equilibria and classify each as {stable, neutrally stable, unstable} and {node, saddle point, spiral/loop}. *(5 points)*

The equilibria are located at $(0,1)$ and $(0,-1)$. Linearizing about $(0, y_0)$,

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} &= \begin{pmatrix}\frac{\partial}{\partial x}(xy) & \frac{\partial}{\partial y}(xy) \\ \frac{\partial}{\partial x}(y^2-1) & \frac{\partial}{\partial y}(y^2-1)\end{pmatrix}_{x=0,y=x_0}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} \\
&= \begin{pmatrix} y & x \\ 0 & 2y\end{pmatrix}_{x=0,y=x_0}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} \\
&= \begin{pmatrix} y_0 & 0 \\ 0 & 2y_0\end{pmatrix}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix}
\end{aligned}$$

The eigenvalues are $\lambda = y_0$ and $\lambda = 2y_0$. Thus, the equilibrium $(0,1)$, with $y_0 = 1$, is an unstable node, and the equilibrium $(0,-1)$, with $y_0 = -1$, is a stable node.

(b) Plot the phase plane for this model. Be sure to include the equilibria, isoclines with arrows, and sample trajectories. *(5 points)*