Math 142-2, Homework 8

Your name here

May 27, 2014

Problem 77.1

If $u = u_{\max}(1 - \rho/\rho_{\max})$, then what is the velocity of a traffic shock separating densities ρ_0 and ρ_1? (Simplify the expression as much as possible.) Show that the shock velocity is the average of the density wave velocities associated with ρ_0 and ρ_1.

Your solution goes here

Problem 77.2

If $u = u_{\max}(1 - \rho^2/\rho_{\max}^2)$, then what is the velocity of a traffic shock separating densities ρ_0 and ρ_1? (Simplify the expression as much as possible.) Show that the shock velocity is not the average of the density wave velocities associated with ρ_0 and ρ_1.

Your solution goes here

Problem 77.3

A weak shock is a shock in which the shock strength (the difference in densities) is small. For a weak shock, show that the shock velocity is approximately the average of the density wave velocities associated with the two densities. [Hint: Use Taylor series methods.]

Your solution goes here

Problem 78.2

Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$ and that the initial traffic density is

$$\rho(x,0) = \begin{cases} \frac{\rho_{\max}}{5} & x < 0 \\ \frac{3\rho_{\max}}{5} & x > 0. \end{cases}$$

(a) Sketch the initial density.

Your solution goes here

(b) Determine and sketch the density at later times.

Your solution goes here

(c) Determine the path of a car (in space-time) which starts at $x = -x_0$ (behind $x = 0$).

Your solution goes here

(d) Determine the path of a car (in space-time) which starts at $x = x_0$ (ahead of $x = 0$).

Your solution goes here

Problem 78.3

Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$ and at $t = 0$, the traffic density is

$$\rho(x, 0) = \begin{cases} \frac{\rho_{\max}}{3} & x < 0 \\ \frac{2\rho_{\max}}{3} & x > 0. \end{cases}$$

Why does the density not change in time?

Your solution goes here

Problem 78.4

Referring to the problem in Sec. 78, show algebraically that the value of the shock velocity is between the velocities of the two density waves.

Your solution goes here

Problem 79.2

Suppose that

$$\rho(x, 0) = \begin{cases} \rho_0 & x > 0 \\ 0 & x < 0 \end{cases}$$

Determine the velocity of the shock. Briefly give a physical explanation of the result.

Your solution goes here

Problem 79.3

The initial traffic density on a road is

$$\rho(x,0) = \begin{cases} 0 & x \le 0 \\ \frac{\rho_{\max} x}{L} & 0 < x < L \\ \rho_{\max} & x \ge L \end{cases}$$

Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$**.**

(a) Sketch the initial density.

Your solution goes here

(b) Show that all characteristics from the interval $0 < x < L$ **(and** $t = 0$**) intersect at the point** $x = L/2$**,** $t = L/(2u_{\max})$**.**

Your solution goes here

(c) A traffic shock will form at this point. Find its subsequent motion.

Your solution goes here

(d) Sketch the $x - t$ **plane, showing the shock and the characteristics necessary to determine** $\rho(x,t)$**.**

Your solution goes here

(e) Sketch $\rho(x,t)$ **before and after the shock.**

Your solution goes here

(f) Describe briefly how the individual automobiles behave (do not determine their paths mathematically).

Your solution goes here

Problem 80.1

Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$.

(a) Show that the time of intersection of neighboring characteristics (corresponding to the collision of two observers initially at x_1 and x_2 moving with constant density ρ_1 and ρ_2) is

$$t = \frac{\rho_{\max}}{2u_{\max}\frac{\Delta\rho}{\Delta x}},$$

where $\Delta x = x_2 - x_1$ and $\Delta\rho = \rho_2 - \rho_1$.

Your solution goes here

(b) Extend this result to the limit as $x_2 \to x_1$ to determine when a shock will form from the characteristics that originate in the vicinity of some location x_1.

Your solution goes here

(c) If at $t = 0$,

$$\rho(x, 0) = \rho_{\max} \exp\left(-\frac{x^2}{L^2}\right).$$

(1) Sketch the initial density.

Your solution goes here

(2) Determine the time of the first shock.

Your solution goes here

(3) Where does this shock first occur?

Your solution goes here

Problem 82.1

Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$. If the initial density is

$$\rho(x, 0) = \begin{cases} \rho_1 & x < 0 \\ \rho_2 & a > x > 0 \\ \rho_3 & x > a \end{cases}$$

with $0 < \rho_1 < \rho_2 < \rho_3 < \rho_{\max}$, then determine the density at later times. [Hint: See exercise 77.1. Calculate the shock between ρ_1 and ρ_2. Show that this shock moves faster than the shock between ρ_2 and ρ_3. What happens after these two shocks meet?]

Your solution goes here

Problem 82.2

Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$ and that the initial traffic density is

$$\rho(x,0) = \begin{cases} \rho_1 & |x| > a \\ \rho_0 & |x| < a \end{cases}$$

where $\rho_1 > \rho_0$. Determine the density at later times.

Your solution goes here

Additional Problem 1

Let $N(a,t)$ be the number of individuals with age a at time t, which is valid for $a \geq -g$ and $t \geq 0$, where g is the gestation time (time between conception and birth) is g. The conception rate $b(a)$ and death rate $c(a)$ are functions of age. The initial age distribution is $f(a)$. In the previous homework assignment, we worked out a continuous model for this, which took the form

$$\begin{aligned} \frac{\partial N}{\partial t} + \frac{\partial N}{\partial a} &= -Nc & & a > -g, t > 0 \\ N(-g,t) &= \int_{-g}^{\infty} N(a,t)b(a)\,da & & t > 0 \\ N(a,0) &= f(a) & & a > -g \end{aligned}$$

(a) What are the characteristics of this PDE?

Your solution goes here

(b) Let M_0 be the number of individuals who are the same age a_0 at time t_0. After some time $T = t_1 - t_0$ has passed, the individuals in this group are now of age $a_1 = a_0 + T$, and there are now M_1 of them still alive. Show that $M_1 = M_0 e^{-rT}$, where r is the average death rate for individuals of the age range $[a_0, a_1]$.

Your solution goes here

(c) Show that with this model if births equal deaths

$$\int_{-g}^{\infty} N(a,t)b(a)\,da = \int_{-g}^{\infty} N(a,t)c(a)\,da$$

then the total population $T(t)$ does not change. You may assume that $N(a,t) \to 0$ as $a \to \infty$.

Your solution goes here