Math 142-2, Homework 2

Your name here

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Problem 35.3

Consider a species in which both no individuals live to three years old and only one-year olds reproduce.

(a) Show that $b_0 = 0$, $b_2 = 0$, $d_2 = 1$ satisfy both conditions.

Your solution goes here

(b) Let $b_1 = b$. What is the A matrix?

Your solution goes here

(c) Determine (a computer is not necessary) the time development of a population initially consisting of $N_0 = 100$, $N_1 = 100$, $N_2 = 100$. Explicitly calculate the following three cases of birth and death rates:

1. $b = 1, d_0 = \frac{1}{2}, d_1 = \frac{1}{2}$
2. $b = 2, d_0 = \frac{1}{2}, d_1 = \frac{1}{2}$
3. $b = 3, d_0 = \frac{1}{2}, d_1 = \frac{1}{2}$

Your solution goes here

Problem 37.2

F.E. Smith suggested a different simple model of the population growth of a species limited by the food supply based on experiments on a type of water bug. As in the logistic model, the growth rate is proportional to the difference between the available food f_a and the subsistence level of food consumption f_c:

$$\frac{1}{N}\frac{dN}{dt} = \alpha(f_a - f_c).$$

However, previously f_c was assumed proportional to the number in individuals of the species. Smith instead assumed that more food is necessary for survival during the growing phase of a population. Consequently a simple model would be

$$f_c = \beta N + \gamma \frac{dN}{dt}$$

with $\gamma > 0$. What differential equation describes this model? What are the equilibrium populations?

Your solution goes here

Problem 37.4

Which of the following are reasonable models of the spread of a disease among a finite number of people:
(1) $dN/dt = \alpha N$
(2) $dN/dt = \alpha(N_T - N)$
(3) $dN/dt = \alpha(N - N_T)$
where N is the number of infected individuals and N_T is the total population.

Your solution goes here

Problem 37.5

A certain species has an instantaneous growth rate of 27 percent per year when not affected by crowding. Experimentally, for each 1000 of the species the birth rate drops by 12 per 1000 per year, and the death rate increases by 50 per 1000 per year. Determine the parameters of a logistic equation which models this species. What is the expected nonzero equilibrium population?

Your solution goes here

Problem 38.3

Consider Smith's model of population growth (see exercise 37.2)

$$\frac{dN}{dt} = \frac{N\alpha(f_a - \beta N)}{1 + N\alpha\gamma}.$$

Are the equilibrium populations stable?

Your solution goes here

Problem 39.1

Consider $dN/dt = \alpha N^2 - \beta N$ (with $\alpha > 0$, $\beta > 0$).

(a) How does the growth rate depend on the population?

Your solution goes here

(b) Sketch the solution in the phase plane.

Your solution goes here

(c) Obtain the exact solution.

Your solution goes here

(d) Show how both parts (b) and (c) illustrate the following behavior:
(i) If $N_0 > \beta/\alpha$, then $N \to \infty$. (At what time does $N \to \infty$?)

Your solution goes here

(ii) If $N_0 < \beta/\alpha$, then $N \to 0$.

Your solution goes here

(iii) What happens if $N_0 = \beta/\alpha$?

Your solution goes here

Problem 39.2

The general solution of the logistic equation, equation 39.1, has been shown to be equation 39.3. Show that this exact solution has the following properties: (1) It is defined for all $t \geq 0$.

Your solution goes here

(2) As $t \to \infty$, $N \to a/b$ for all initial conditions except $N_0 = 0$. (Note: It takes an infinite amount of time to reach the equilibrium population.)

Your solution goes here

Problem 39.4

Consider the following growth model:

$$\frac{dN}{dt} = aN + bN^2 \qquad \text{with } a > 0,\ b > 0.$$

(a) How does the growth rate depend on N?

Your solution goes here

(b) What would you expect happens to the population?

Your solution goes here

(c) By using the phase plane, show that the population tends towards infinity.

Your solution goes here

(d) By considering the exact solution, show that the population reaches infinity in finite time; what might be called a population explosion.

Your solution goes here

Additional Problem

Consider the population model $dN/dt = -(a - bN)^2$, where $a > 0$, $b > 0$, and $N > 0$. (At $N = 0$, the population is extinct, and $dN/dt = 0$.)

(a) Show that this model is just the logistic equation with a constant migration term added to it. Are individuals migrating into the population or away from it?

Your solution goes here

(b) Sketch the solution in the phase plane.

Your solution goes here

(c) Identify the equilibrium. Is it stable? Justify your answer.

Your solution goes here

(d) Find the exact solution in terms of a, b, and the initial value N_0.

Your solution goes here

(e) Show that if N_0 is greater than the equilibrium value, N approaches equilibrium but neither reaches it nor crosses it.

Your solution goes here

(f) Show that if N_0 is less than the equilibrium value, N becomes extinct in finite time.

Your solution goes here