Math 142-2, Homework 1

Your name here

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Problem 34.4

Suppose the growth rate of a certain species is not constant, but depends in a known way on the temperature of its environment. If the temperature is known as a function of time, derive an expression for the future population (which is initially N_0). Show that the population grows or decays with an exponential growth coefficient, ($R_E(t)$, where $N(t) \propto e^{R_E(t)t}$), equal to the average of the time-dependent growth rate.

Your solution goes here

Problem 34.5

In this problem we study the effect of any time-dependent migration, $f(t)$. Consider both the resulting discrete and continuous growth models:

$$N_{m+1} - N_m = R_0 \Delta t N_m + \Delta t f_m \tag{1}$$

$$\frac{dN}{dt} = R_0 N + f(t) \tag{2}$$

(a) If the growth rate is zero ($R_0 = 0$), show that the solution of the discrete growth model is analogous to integration.

Your solution goes here

(b) If the growth rate is nonzero, show that

$$N_{m+1} = \alpha N_m + f_m \Delta t \tag{3}$$

where $\alpha = 1 + R_0 \Delta t$. By explicit calculation of $N_1, N_2, N_3, \ldots$, determine N_m, if N_0 is known.

Your solution goes here

(c) For the differential equation (2), calculate the solution using the integrating factor $e^{-R_0 t}$.

Your solution goes here

(d) In this part of the problem, we wish to develop a technique to solve difference equation (1), analogous to the integrating factor method used in part (c). By dividing equation (3) by α^{m+1}, show that

$$Q_{m+1} - Q_m = \frac{f_m}{\alpha^{m+1}} \Delta t \tag{4}$$

where $Q_n = N_m/\alpha^m$. Using the result of part (a), solve equation (4), and thus determine N_m. Show that your answer agrees with part (b), and show how your answer is analogous to the results of part (c).

Your solution goes here

Problem 34.9

Suppose that one species has an instantaneous growth rate of α percent per year while another species grows in discrete units of time at the annual rate of β percent per year. Suppose the second species has four growth periods a year. What relationship exists between α and β if both species (starting with the same number) have the same number 5 years later?

Your solution goes here

Problem 34.14

The parameters of a theoretical population growth curve are ofter estimated making the best fit of this curve to some data. If discrete population data is known (not necessarily measured at equal time intervals), $N_d(t_m)$, then the mean-square deviation between the data and a theoretical curve, $N(t)$, is

$$\sum_m [N(t_m) - N_d(t_m)]^2,$$

the sum of the squared differences. The "best" fit is often defined as those values which minimize the above mean-square deviation.

(a) Assume that the initial population is known with complete certainty, so that we insist that the theoretical population curve initially agree exactly. Assume the theoretical curve exhibits exponential growth. By minimizing the above mean-square deviation, obtain an equation for the best estimate of the growth rate. Show that this is a transcendental equation.

Your solution goes here

(b) One way to bypass the difficulty in part (a) is to fit the natural logarithm of the data

to the natural logarithm of the theoretical curve. In this way, the mean-square deviation is

$$\sum_m [\ln N_0 + R_0 t_m - \ln N_d(t_m)]^2,$$

Show that this method now is the least squares fit of a straight line to data. If N_0 is known ($N_0 = N_d(t_0)$), determine the best estimate of the growth rate using this criteria.

Your solution goes here

(c) Redo part (b) assuming that best estimate of the initial population is also desired (i.e., minimize the mean-square deviation with respect to both N_0 and R_0).

Your solution goes here

Problem 34.17

In this section it was shown that the difference equation of discrete growth becomes a differential equation in the limit as $\Delta t \to 0$. If a differential equating is known, for example,

$$\frac{dN(t)}{dt} = R_0 N(t),$$

we will show how to calculate a difference equation which corresponds to it. The idea is useful for obtaining numerical solutions on the computer of differential equations. If a Taylor series is used, then

$$N(t + \Delta t) = N(t) + \Delta t \frac{dN}{dt} + O((\Delta t)^2).$$

Thus, if higher order terms are neglected:

(a) Show that

$$\frac{dN}{dt} = \frac{N(t + \Delta t) - N(t)}{\Delta t} + O(\Delta t).$$

Replacing the derivative by this difference is known as Euler's method to numerically solve ordinary differential equations.

Your solution goes here

(b) Show that the resulting difference equation (called the discretization of the differential equation) is itself a discrete model of population growth.

Your solution goes here

(c) At the very least, for this to be a reasonable procedure, the first neglected term in the Taylor series $(\Delta t)^2 d^2N/dt^2/2$ must be much less than $\Delta t(dN/dt)$. Using the differential equation, are there thus any restrictions on the discretization time Δt for this problem?

Your solution goes here