Math 142-2, Midterm

Name: ________________________________ ID: ______________

Problem 1

Consider a damped spring given by the equation $mx'' + cx'|x'| + kx = 0$.

(a) Show that total energy can never increase. Can it decrease?

(b) Why is $c(x')^2$ not used for the damping term?

(c) What are the units of c?

Problem 1 (continued)

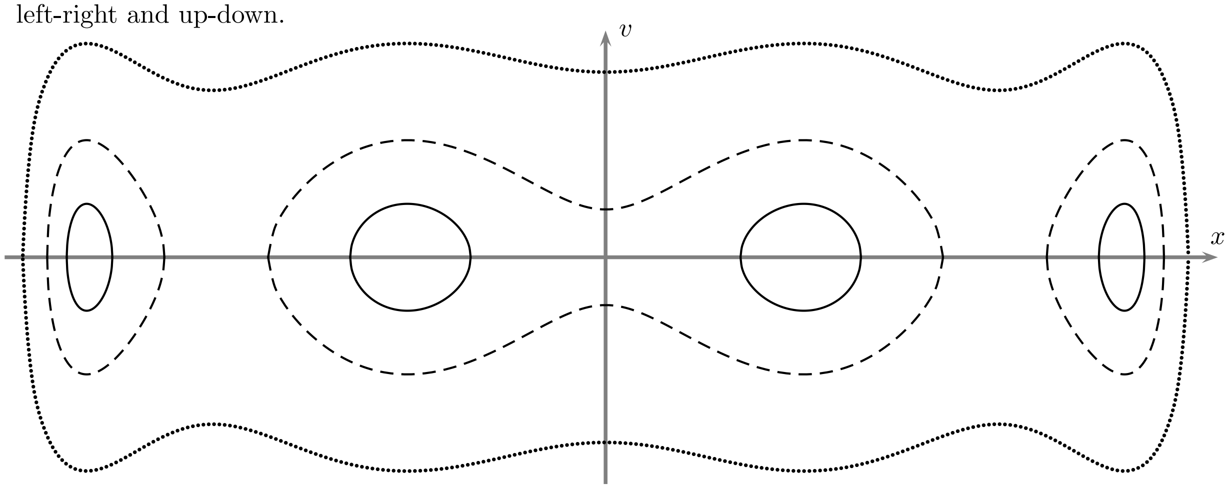
Consider a damped spring given by the equation $mx'' + cx'|x'| + kx = 0$.

(d) Determine using linearized stability analysis whether the system is stable, unstable, or neutrally stable.

(e) Is the system stable, unstable, or neutrally stable? Why?

Problem 2

Consider the ODE $mx'' = f(x)$ for a particle, where the force $f(x)$ has the potential energy function $\phi(x)$. Below is part of the phase plane diagram for the resulting ODE. The phase plane is symmetrical left-right and up-down.



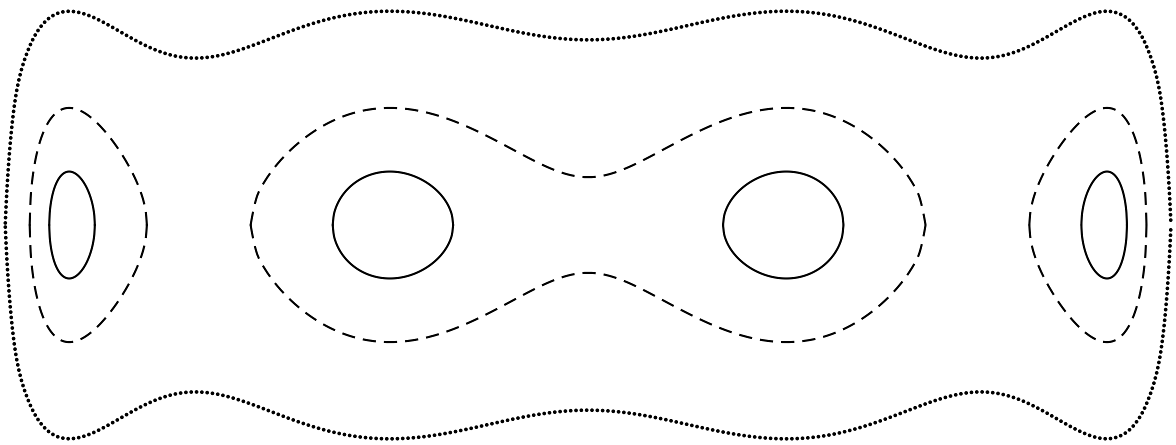
(a) The phase plane shows three energy levels: dotted, dashed, and solid. Which of these corresponds to the highest energy level? Which corresponds to the lowest energy level?

(b) On the phase plane diagram above, mark the stable equilibria with "•" and the unstable equilibria with "∘".

(c) On the phase plane diagram above, sketch the curves whose energy matches the energy of the unstable equilibria. These energy curves may contain more than one piece; be sure to sketch all of them.

(d) Put arrows on all of the curves (including the ones you drew in part (c)) to show the trajectories.

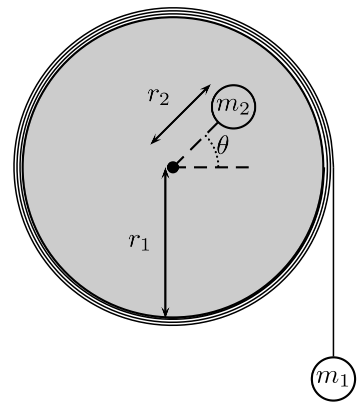
Problem 2 (continued)



(e) Sketch the potential energy function. Show on your plot the energy levels corresponding to the three curves in the phase plane.

Problem 3

A pulley of radius r_1 has wrapped around it a long cable with an object of mass m_1 hanging from it. Another object of mass m_2 is attached to the pulley at a distance of r_2 from the pulley's center. Let θ be the polar angle the attached mass. Assume the cable is arbitrarily long.



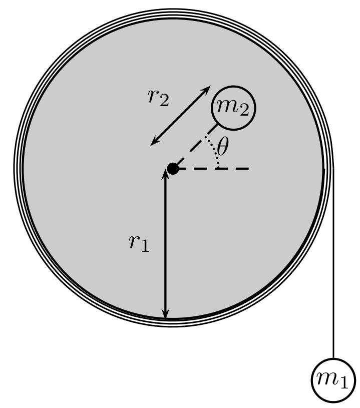
(a) What is the potential energy of the system (in terms of θ)?

(b) What is the total energy of the system (in terms of θ and $\dot{\theta}$)?

(c) Show that this system obeys the ODE

$$(m_1 r_1^2 + m_2 r_2^2)\ddot{\theta} + r_2 m_2 g \cos\theta + r_1 m_1 g = 0.$$

Problem 3 (continued)



(d) If $m_2 < M_e$, for some critical mass M_e, then this system has no equilibria. Find M_e.

(e) If $m_2 < M_e$, describe qualitatively the dynamical behavior of the system.