Tech report: the affine Particle-in-Cell method

Chenfanfu Jiang[†] Craig Schroeder[†] Andrew Selle[⋆] Joseph Teran[⋆†] Alexey Stomakhin[⋆]

[†]University of California Los Angeles [⋆]Walt Disney Animation Studios

1 Introduction

This document presents the proofs that are related to the Affine Particle-in-Cell method. Section 2 describes the notations. Section 3 lists some preliminaries for the proofs. Section 4 proves the following properties of Rigid Particle-in-Cell (RPIC): 1. Preservation of rigid motion velocity field (4.1), 2. Conservation of linear momentum (4.2), 3. Conservation of angular momentum (4.3). Section 5 proves the following properties of Affine Particle-in-Cell (APIC): 1. Preservation of affine velocity field (5.1), 2. Conservation of linear momentum (5.2), 3. Conservation of angular momentum (5.3).

2 Notation

We begin by laying out the notation that we use in this document. Many quantities have subscripts (m_p), and some have superscripts as well ($\boldsymbol{v}_p^n$). Subscripts p and q are used to refer to particles, and subscripts i and j are used to refer to regular grid indices. The superscript distinguishes quantities available or computed at the beginning of the time step ($\boldsymbol{v}_p^n$) from quantities that are computed for use at the beginning of the next time step ($\boldsymbol{v}_p^{n+1}$). We use lowercase bold for vectors ($\boldsymbol{v}_p^n$) and uppercase bold for matrices ($\boldsymbol{B}_p^n$), with the exception of angular momentum, a vector quantity that is normally denoted by $\boldsymbol{L}_p^n$. The notation we use is summarized in Table 1.

3 Preliminaries

When we consider conservation of angular momentum when transferring from grid to particles at the end of a time step, we need to consider angular momentum to be defined over moved grid nodes and we use the notation $\tilde{\boldsymbol{x}}_i^{n+1} = \boldsymbol{x}_i + \Delta t \tilde{\boldsymbol{v}}_i^{n+1}$ to indicate this. To avoid confusion, rather than referring to unmoved grid nodes at the beginning of the time step as $\boldsymbol{x}_i$, we will use $\boldsymbol{x}_i^n$ to emphasize that they have not been dynamically updated yet, whereas the $\tilde{\boldsymbol{x}}_i^{n+1}$ have been. We adopt this notation in the remainder of the document.
We will also use a few properties of standard interpolating functions, namely.

$$\sum_i w_{ip}^n = 1$$

$$\sum_i w_{ip}^n \boldsymbol{x}_i^n = \boldsymbol{x}_p^n$$

$$\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) = \boldsymbol{0}$$

Here $w_{ip}^n = N_i(\boldsymbol{x}_p^n)$ are the weights at time t^n, where $N_{\mathbf{i}}(\boldsymbol{x})$ is the interpolation function associated with grid node i.
The definitions of linear and angular momentum on the grid as well as linear momentum on particles are listed below. The angular momentum on particles is written in different forms for RPIC and APIC and will be defined in the corresponding sections.

Definition 3.1. *The total linear momentum on the grid (after the particle-to-grid transfer at time n) is*

$$\boldsymbol{p}_{tot}^{G,n} = \sum_i m_i^n \boldsymbol{v}_i^n$$

Definition 3.2. *The total angular momentum on the grid (after the particle-to-grid transfer at time n) is*

$$\boldsymbol{L}_{tot}^{G,n} = \sum_i \boldsymbol{x}_i^n \times m_i^n \boldsymbol{v}_i^n$$

Definition 3.3. *The total linear momentum on the grid (after the grid dynamics, before the grid-to-particle transfer in the end of time n) is*

$$\boldsymbol{p}_{tot}^{G,n+1} = \sum_i m_i^n \tilde{\boldsymbol{v}}_i^{n+1}$$

Definition 3.4. *The total angular momentum on the grid (after the grid dynamics, before the grid-to-particle transfer in the end of time n) is*

$$\boldsymbol{L}_{tot}^{G,n+1} = \sum_i \tilde{\boldsymbol{x}}_i^{n+1} \times m_i^n \tilde{\boldsymbol{v}}_i^{n+1}$$

Definition 3.5. *The total linear momentum on particles (before the particle-to-grid transfer at time n) is*

$$\boldsymbol{p}_{tot}^{P,n} = \sum_p m_p \boldsymbol{v}_p^n$$

Symbol	location	type	meaning
m_p	p	s	mass
$\boldsymbol{x}_p^n$	p	v	position
$\boldsymbol{v}_p^n$	p	v	velocity
$\boldsymbol{L}_p^n$	p	v	angular momentum
$\boldsymbol{K}_p^n$	p	m	inertia tensor
$\boldsymbol{B}_p^n$	p	m	affine state
$\boldsymbol{D}_p^n$	p	m	inertia-like tensor
m_i^n	n	s	mass
$\boldsymbol{x}_i^n$	n	v	position
$\tilde{\boldsymbol{x}}_i^{n+1}$	n	v	moving grid position
$\boldsymbol{v}_i^n$	n	v	velocity
$\tilde{\boldsymbol{v}}_i^{n+1}$	n	v	intermediate velocity
w_{ip}^n	n+p	s	weights
∇w_{ip}^n	n+p	v	weight gradients
$\boldsymbol{p}_{tot}^{P,n}$	g	v	total particle linear momentum
$\boldsymbol{p}_{tot}^{G,n}$	g	v	total grid linear momentum
$\boldsymbol{L}_{tot}^{P,n}$	g	v	total particle angular momentum
$\boldsymbol{L}_{tot}^{G,n}$	g	v	total grid angular momentum
$\boldsymbol{v}^*$	g	m	cross product matrix of $\boldsymbol{v}$
$\boldsymbol{\epsilon}$	g	t	permutation tensor
Δt	g	s	time step size

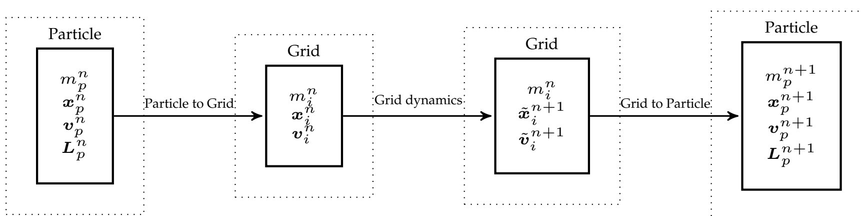
Table 1: Summary of notation used in this paper. Locations are p (particle), n (regular grid node), or g (global; does not live at any location in space). Quantities are of type s (scalar), v (vector), m (matrix) or t (rank-3 tensor).

Definition 3.6. *The total linear momentum on particles (after the grid-to-particle transfer in the end of time n) is*

$$\boldsymbol{p}_{tot}^{P,n+1} = \sum_p m_p \boldsymbol{v}_p^{n+1}$$

4 Piecewise rigid

Here is a data flow diagram for Rigid Particle-in-Cell method.



RPIC stores mass m_p, position $\boldsymbol{x}_p$, velocity $\boldsymbol{v}_p$ and angular momentum $\boldsymbol{L}_p$ on particles. The transfer from particles to the grid are given by

$$\boxed{\begin{aligned} m_i^n &= \sum_p w_{ip}^n m_p \\ \boldsymbol{K}_p^n &= \sum_j w_{jp}^n m_p (\boldsymbol{x}_j^n - \boldsymbol{x}_p^n)^* (\boldsymbol{x}_j^n - \boldsymbol{x}_p^n)^{*T} \\ m_i^n \boldsymbol{v}_i^n &= \sum_p w_{ip}^n m_p (\boldsymbol{v}_p^n + ((\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n) \times (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)). \end{aligned}} \tag{1}$$

One may imagine this transfer as distributing the masses $w_{ip}^n m_p$ from a rigid body to the grid node i. $\boldsymbol{K}_p$ is the inertia tensor associated with the local rigid body represented by particle p. For any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{u}^*$ is defined to be the cross product matrix of $\boldsymbol{u}$, so that $\boldsymbol{u}^*\boldsymbol{v} = \boldsymbol{u} \times \boldsymbol{v}$ and $(\boldsymbol{u}^*)^T \boldsymbol{v} = \boldsymbol{v} \times \boldsymbol{u}$.
The transfer from the grid to particles are given by

$$\boxed{\begin{aligned} \boldsymbol{v}_p^{n+1} &= \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} \\ \boldsymbol{L}_p^{n+1} &= \sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \times m_p \tilde{\boldsymbol{v}}_i^{n+1}. \end{aligned}} \tag{2}$$

4.1 Preservation of rigid motion velocity field

Proposition 4.1. *Let $\Delta t = 0$ and consider the the process of transferring grid velocity ($\tilde{\boldsymbol{v}}_i^{n+1}$) information to particles ($\boldsymbol{v}_p^{n+1}$, $\boldsymbol{L}_p^{n+1}$) and then back to the grid ($\boldsymbol{v}_i^{n+1}$) with the Rigid Particle-in-Cell method. If the velocities before the transfer represent rigid motion, $\tilde{\boldsymbol{v}}_i^{n+1} = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_i^n$, where $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are any constant vectors, then after the process, this velocity field is exactly reproduced: $\boldsymbol{v}_i^{n+1} = \tilde{\boldsymbol{v}}_i^{n+1}$.*

Proof. Since $\Delta t = 0$, we have $w_{ip}^n = w_{ip}^{n+1}$ and $\boldsymbol{x}_p^n = \boldsymbol{x}_p^{n+1}$, so that $\boldsymbol{K}_p^n = \boldsymbol{K}_p^{n+1}$ and $m_i^n = m_i^{n+1}$. The grid to particle transfer is (for $\boldsymbol{v}_p$)

$$\begin{aligned}
\boldsymbol{v}_p^{n+1} &= \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_i w_{ip}^n (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_i^n) \\
&= \sum_i w_{ip}^n \boldsymbol{v} + \sum_i w_{ip}^n \boldsymbol{\omega} \times \boldsymbol{x}_i^n \\
&= \boldsymbol{v} \sum_i w_{ip}^n + \boldsymbol{\omega} \times \sum_i w_{ip}^n \boldsymbol{x}_i^n \\
&= \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_p^n,
\end{aligned}$$

and for $\boldsymbol{L}_P$:

$$\begin{aligned}
\boldsymbol{L}_p^{n+1} &= \sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \times m_p \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \times m_p (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_i^n) \\
&= \left(\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \right) \times m_p \boldsymbol{v} + \sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \times m_p (\boldsymbol{\omega} \times \boldsymbol{x}_i^n) \\
&= \sum_i m_p w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^* (\boldsymbol{x}_i^n)^{*T} \boldsymbol{\omega} \\
&= \sum_i m_p w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^* (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^{*T} \boldsymbol{\omega} + \sum_i m_p w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^* (\boldsymbol{x}_p^n)^{*T} \boldsymbol{\omega} \\
&= \boldsymbol{K}_p^n \boldsymbol{\omega} + m_p \left(\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \right)^* (\boldsymbol{x}_p^n)^{*T} \boldsymbol{\omega} \\
&= \boldsymbol{K}_p^n \boldsymbol{\omega}
\end{aligned}$$

A particle to grid transfer is then performed:

$$\begin{aligned}
m_i^{n+1} \boldsymbol{v}_i^{n+1} &= \sum_p w_{ip}^{n+1} m_p (\boldsymbol{v}_p^{n+1} + ((\boldsymbol{K}_p^{n+1})^{-1} \boldsymbol{L}_p^{n+1}) \times (\boldsymbol{x}_i^{n+1} - \boldsymbol{x}_p^{n+1})) \\
m_i^n \boldsymbol{v}_i^{n+1} &= \sum_p w_{ip}^n m_p (\boldsymbol{v}_p^{n+1} + ((\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^{n+1}) \times (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\
&= \sum_p w_{ip}^n m_p (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_p^n + ((\boldsymbol{K}_p^n)^{-1} \boldsymbol{K}_p^n \boldsymbol{\omega}) \times (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\
&= \sum_p w_{ip}^n m_p (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_p^n + \boldsymbol{\omega} \times (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\
&= \left(\sum_p w_{ip}^n m_p \right) (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_i^n) \\
&= m_i^n (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_i^n)
\end{aligned}$$

We can see $\boldsymbol{v}_i^{n+1} = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}_i^n = \tilde{\boldsymbol{v}}_i^{n+1}$. Therefore, the rigid motion velocity field represented by $\boldsymbol{v}$ and $\boldsymbol{\omega}$ is preserved in the full transfer cycle. □

4.2 Conservation of linear momentum

4.2.1 Particle to grid

Proposition 4.2. *Linear momentum is conserved during the RPIC transfer from particles to the grid. $\boldsymbol{p}_{tot}^{G,n} = \boldsymbol{p}_{tot}^{P,n}$ under transfer 1.*

Proof. The linear momentum on the grid after transferring from particles is

$$
\begin{aligned}
\boldsymbol{p}_{tot}^{G,n} &= \sum_i m_i^n \boldsymbol{v}_i^n \\
&= \sum_i \left(\sum_p w_{ip}^n m_p (\boldsymbol{v}_p^n + ((\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n) \times (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \right) \\
&= \sum_{i,p} w_{ip}^n m_p \boldsymbol{v}_p^n + \sum_{i,p} w_{ip}^n m_p (((\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n) \times (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\
&= \sum_p \left(\sum_i w_{ip}^n \right) m_p \boldsymbol{v}_p^n + \sum_p m_p ((\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n) \times \left(\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \right) \\
&= \sum_p m_p \boldsymbol{v}_p^n \\
&= \boldsymbol{p}_{tot}^{P,n}
\end{aligned}
$$

□

4.2.2 Grid to particle

Proposition 4.3. *Linear momentum is conserved during the RPIC transfer from the grid to particles.* $\boldsymbol{p}_{tot}^{P,n+1} = \boldsymbol{p}_{tot}^{G,n+1}$ *under transfer 2.*

Proof. The linear momentum on the particles after transferring from the grid is

$$
\begin{aligned}
\boldsymbol{p}_{tot}^{P,n+1} &= \sum_p m_p \boldsymbol{v}_p^{n+1} \\
&= \sum_p m_p \left(\sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} \right) \\
&= \sum_i \left(\sum_p w_{ip}^n m_p \right) \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_i m_i^n \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \boldsymbol{p}_{tot}^{G,n+1}
\end{aligned}
$$

□

4.3 Conservation of angular momentum

With $m_p, \boldsymbol{x}_p, \boldsymbol{v}_p$ and $\boldsymbol{L}_p$, we can define the total angular momentum on particles in the piecewise rigid case.

Definition 4.1. *The total angular momentum on particles (before the particle-to-grid transfer at time n) represented by RPIC is*

$$\boldsymbol{L}_{tot}^{P,n} = \sum_p (\boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \boldsymbol{L}_p^n)$$

Definition 4.2. *The total angular momentum on particles (after the grid-to-particle transfer in the end of time n) represented by RPIC is*

$$\boldsymbol{L}_{tot}^{P,n+1} = \sum_p (\boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} + \boldsymbol{L}_p^{n+1})$$

4.3.1 Particle to grid

Proposition 4.4. *Angular momentum is conserved during the RPIC transfer from particles to the grid.* $\boldsymbol{L}_{tot}^{G,n} = \boldsymbol{L}_{tot}^{P,n}$ *under transfer 1.*

Proof. The angular momentum on the grid after transferring from particles is

$$\begin{aligned}
\boldsymbol{L}_{tot}^{G,n} &= \sum_i \boldsymbol{x}_i^n \times m_i^n \boldsymbol{v}_i^n \\
&= \sum_i \boldsymbol{x}_i^n \times \left(\sum_p w_{ip}^n m_p (\boldsymbol{v}_p^n + ((\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n) \times (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \right) \\
&= \sum_{i,p} \boldsymbol{x}_i^n \times w_{ip}^n m_p \boldsymbol{v}_p^n + \sum_{i,p} \boldsymbol{x}_i^n \times w_{ip}^n m_p (((\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n) \times (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\
&= \sum_p \left(\sum_i w_{ip}^n \boldsymbol{x}_i^n \right) \times m_p \boldsymbol{v}_p^n + \sum_{i,p} \boldsymbol{x}_i^n \times w_{ip}^n m_p (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^{*T} (\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_{i,p} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \times w_{ip}^n m_p (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^{*T} (\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n + \sum_{i,p} \boldsymbol{x}_p^n \times w_{ip}^n m_p (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^{*T} (\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p \left(\sum_i m_p w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^* (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^{*T} \right) (\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n + \sum_p \boldsymbol{x}_p^n \times m_p \left(\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \right)^{*T} (\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p \boldsymbol{K}_p^n (\boldsymbol{K}_p^n)^{-1} \boldsymbol{L}_p^n \\
&= \sum_p (\boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \boldsymbol{L}_p^n) \\
&= \boldsymbol{L}_{tot}^{P,n}
\end{aligned}$$

□

4.3.2 Grid to particle

Proposition 4.5. *Angular momentum is conserved during the RPIC transfer from the grid to particles.* $\boldsymbol{L}_{tot}^{P,n+1} = \boldsymbol{L}_{tot}^{G,n+1}$ *under transfer 2.*

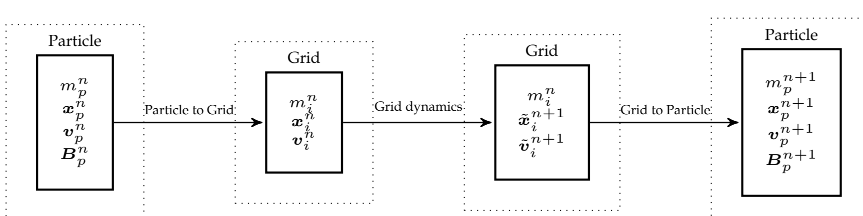
Proof. The angular momentum on the particles after transferring from the grid is

$$\begin{aligned}
\boldsymbol{L}_{tot}^{P,n+1} &= \sum_p (\boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} + \boldsymbol{L}_p^{n+1}) \\
&= \sum_p \left(\boldsymbol{x}_p^{n+1} \times m_p \left(\sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} \right) + \left(\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \times m_p \tilde{\boldsymbol{v}}_i^{n+1} \right) \right) \\
&= \sum_{i,p} \left(\boldsymbol{x}_p^{n+1} \times m_p w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} + w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \times m_p \tilde{\boldsymbol{v}}_i^{n+1} \right) \\
&= \sum_{i,p} (\boldsymbol{x}_p^{n+1} - \boldsymbol{x}_p^n) \times m_p w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} + \sum_{i,p} w_{ip}^n \boldsymbol{x}_i^n \times m_p \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \Delta t \sum_p \boldsymbol{v}_p^{n+1} \times m_p \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} + \sum_{i,p} w_{ip}^n (\tilde{\boldsymbol{x}}_i^{n+1} - \Delta t \tilde{\boldsymbol{v}}_i^{n+1}) \times m_p \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \Delta t \sum_{i,p} \boldsymbol{v}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} + \sum_{i,p} w_{ip}^n \tilde{\boldsymbol{x}}_i^{n+1} \times m_p \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_i \left(\sum_p w_{ip}^n m_p \right) \tilde{\boldsymbol{x}}_i^{n+1} \times \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_i \tilde{\boldsymbol{x}}_i^{n+1} \times m_i^n \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \boldsymbol{L}_{tot}^{G,n+1}
\end{aligned}$$

□

5 Piecewise Affine

Here is a data flow diagram for Affine Particle-in-Cell method.



APIC stores mass m_p, position $\boldsymbol{x}_p$, velocity $\boldsymbol{v}_p$ and matrix $\boldsymbol{B}_p$ on particles. The transfer from particles to the grid are given by

$$\boxed{\begin{aligned} m_i^n &= \sum_p w_{ip}^n m_p \\ \boldsymbol{D}_p^n &= \sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)(\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T = \sum_i w_{ip}^n \boldsymbol{x}_i^n (\boldsymbol{x}_i^n)^T - \boldsymbol{x}_p^n (\boldsymbol{x}_p^n)^T \\ m_i^n \boldsymbol{v}_i^n &= \sum_p w_{ip}^n m_p (\boldsymbol{v}_p^n + \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \end{aligned}} \tag{3}$$

with the transfer to particles given by

$$\boxed{\begin{aligned} \boldsymbol{v}_p^{n+1} &= \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} \\ \boldsymbol{B}_p^{n+1} &= \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T. \end{aligned}} \tag{4}$$

5.1 Preservation of affine velocity fields

Proposition 5.1. *Let $\Delta t = 0$ and consider the the process of transferring velocity ($\tilde{\boldsymbol{v}}_i^{n+1}$) information to particles ($\boldsymbol{v}_p^{n+1}$, $\boldsymbol{B}_p^{n+1}$) and then back to the grid ($\boldsymbol{v}_i^{n+1}$) with the Affine Particle-in-Cell method. If the velocities before the transfer represent an affine velocity field, $\tilde{\boldsymbol{v}}_i^{n+1} = \boldsymbol{v} + \mathbf{C}\boldsymbol{x}_i^n$, where $\boldsymbol{v}$ is a vector and $\mathbf{C}$ is a matrix, then after the process, this velocity field is exactly reproduced: $\boldsymbol{v}_i^{n+1} = \tilde{\boldsymbol{v}}_i^{n+1}$.*

Proof. Since $\Delta t = 0$, we have $w_{ip}^n = w_{ip}^{n+1}$ and $\boldsymbol{x}_p^n = \boldsymbol{x}_p^{n+1}$, so that $\boldsymbol{D}_p^n = \boldsymbol{D}_p^{n+1}$ and $m_i^n = m_i^{n+1}$. The grid to particle transfer is (for $\boldsymbol{v}_p$)

$$\begin{aligned} \boldsymbol{v}_p^{n+1} &= \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} \\ &= \sum_i w_{ip}^n (\boldsymbol{v} + \mathbf{C}\boldsymbol{x}_i^n) \\ &= \sum_i w_{ip}^n \boldsymbol{v} + \sum_i w_{ip}^n \mathbf{C}\boldsymbol{x}_i^n \\ &= \boldsymbol{v} \sum_i w_{ip}^n + \mathbf{C} \sum_i w_{ip}^n \boldsymbol{x}_i^n \\ &= \boldsymbol{v} + \mathbf{C}\boldsymbol{x}_p^n \end{aligned}$$

and for $\boldsymbol{B}_p$:

$$\begin{aligned} \boldsymbol{B}_p^{n+1} &= \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T \\ &= \sum_i w_{ip}^n (\boldsymbol{v} + \mathbf{C}\boldsymbol{x}_i^n)(\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T \\ &= \sum_i w_{ip}^n \boldsymbol{v} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T + \sum_i w_{ip}^n \mathbf{C}\boldsymbol{x}_i^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T \\ &= \boldsymbol{v} \left(\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \right)^T + \mathbf{C} \sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)(\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T + \sum_i w_{ip}^n \mathbf{C}\boldsymbol{x}_p^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T \\ &= \mathbf{C}\boldsymbol{D}_p^n + \mathbf{C}\boldsymbol{x}_p^n \left(\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \right)^T \\ &= \mathbf{C}\boldsymbol{D}_p^n \end{aligned}$$

A particle to grid transfer is then performed:

$$\begin{aligned} m_i^{n+1} \boldsymbol{v}_i^{n+1} &= \sum_p w_{ip}^{n+1} m_p (\boldsymbol{v}_p^{n+1} + \boldsymbol{B}_p^{n+1} (\boldsymbol{D}_p^{n+1})^{-1} (\boldsymbol{x}_i^{n+1} - \boldsymbol{x}_p^{n+1})) \\ m_i^n \boldsymbol{v}_i^{n+1} &= \sum_p w_{ip}^n m_p (\boldsymbol{v}_p^{n+1} + \boldsymbol{B}_p^{n+1} (\boldsymbol{D}_p^n)^{-1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\ &= \sum_p w_{ip}^n m_p (\boldsymbol{v} + \mathbf{C}\boldsymbol{x}_p^n + \mathbf{C}\boldsymbol{D}_p^n (\boldsymbol{D}_p^n)^{-1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\ &= \sum_p w_{ip}^n m_p (\boldsymbol{v} + \mathbf{C}\boldsymbol{x}_p^n + \mathbf{C}(\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\ &= \left(\sum_p w_{ip}^n m_p \right) (\boldsymbol{v} + \mathbf{C}\boldsymbol{x}_i^n) \\ &= m_i^n (\boldsymbol{v} + \mathbf{C}\boldsymbol{x}_i^n) \end{aligned}$$

We can see $\boldsymbol{v}_i^{n+1} = \boldsymbol{v} + \mathbf{C}\boldsymbol{x}_i^n = \tilde{\boldsymbol{v}}_i^{n+1}$. Therefore, the affine velocity field represented by $\boldsymbol{v}$ and $\mathbf{C}$ is preserved in the full transfer cycle. $\square$

5.2 Conservation of linear momentum

5.2.1 Particle to grid

Proposition 5.2. *Linear momentum is conserved during the APIC transfer from particles to the grid.* $\boldsymbol{p}_{tot}^{G,n} = \boldsymbol{p}_{tot}^{P,n}$ *under transfer 3.*

Proof. The linear momentum on the grid after transferring from particles is

$$\begin{aligned}
\boldsymbol{p}_{tot}^{G,n} &= \sum_i m_i^n \boldsymbol{v}_i^n \\
&= \sum_p \sum_i m_p w_{ip}^n (\boldsymbol{v}_p^n + \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\
&= \sum_{p,i} m_p w_{ip}^n \boldsymbol{v}_p^n + \sum_{p,i} m_p w_{ip}^n \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \\
&= \sum_p m_p \left(\sum_i w_{ip}^n \right) \boldsymbol{v}_p^n + \sum_p m_p \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \left(\sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \right) \\
&= \sum_p m_p \boldsymbol{v}_p^n \\
&= \boldsymbol{p}_{tot}^{P,n}
\end{aligned}$$

□

5.2.2 Grid to particle

Proposition 5.3. *Linear momentum is conserved during the APIC transfer from the grid to particles.* $\boldsymbol{p}_{tot}^{P,n+1} = \boldsymbol{p}_{tot}^{G,n+1}$ *under transfer 4.*

Proof. The linear momentum on the particles after transferring from the grid is

$$\begin{aligned}
\boldsymbol{p}_{tot}^{P,n+1} &= \sum_p m_p \boldsymbol{v}_p^{n+1} \\
&= \sum_p m_p \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_i \left(\sum_p m_p w_{ip}^n \right) \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_i m_i^n \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \boldsymbol{p}_{tot}^{G,n+1}
\end{aligned}$$

□

5.3 Conservation of angular momentum

With $m_p, \boldsymbol{x}_p, \boldsymbol{v}_p$ and $\boldsymbol{B}_p$, we can define the total angular momentum on particles in the piecewise affine case.

Definition 5.1. *The total angular momentum on particles (before the particle-to-grid transfer at time n) represented by APIC is*

$$\boldsymbol{L}_{tot}^{P,n} = \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p m_p (\boldsymbol{B}_p^n)^T : \boldsymbol{\epsilon}$$

Definition 5.2. *The total angular momentum on particles (after the grid-to-particle transfer in the end of time n) represented by APIC is*

$$\boldsymbol{L}_{tot}^{P,n+1} = \sum_p \boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} + \sum_p m_p (\boldsymbol{B}_p^{n+1})^T : \boldsymbol{\epsilon}$$

Note we use the permutation tensor $\boldsymbol{\epsilon}$ in this section. To make these portions easier to read, we take the convention that for any matrix $\boldsymbol{A}$, the contraction $\boldsymbol{A} : \boldsymbol{\epsilon}$ means the same thing as $A_{\alpha\beta}\epsilon_{\alpha\beta\gamma}$. The manipulation $\boldsymbol{u} \times \boldsymbol{v} = (\boldsymbol{v}\boldsymbol{u}^T)^T : \boldsymbol{\epsilon}$ is used to transition from a cross product into the permutation tensor.

5.3.1 Particle to grid

Proposition 5.4. *Angular momentum is conserved during the APIC transfer from particles to the grid.* $\boldsymbol{L}_{tot}^{G,n} = \boldsymbol{L}_{tot}^{P,n}$ *under transfer 3.*

Proof. The angular momentum on the grid after transferring from particles is

$$\begin{aligned}
\boldsymbol{L}_{tot}^{G,n} &= \sum_i \boldsymbol{x}_i^n \times m_i^n \boldsymbol{v}_i^n \\
&= \sum_p \sum_i \boldsymbol{x}_i^n \times m_p w_{ip}^n (\boldsymbol{v}_p^n + \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)) \\
&= \sum_p \sum_i \boldsymbol{x}_i^n \times m_p w_{ip}^n \boldsymbol{v}_p^n + \sum_p \sum_i \boldsymbol{x}_i^n \times m_p w_{ip}^n \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{x}_i^n - \sum_p \left(\sum_i w_{ip}^n \boldsymbol{x}_i^n \right) \times m_p \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{x}_p^n \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p m_p \sum_i w_{ip}^n \boldsymbol{x}_i^n \times (\boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{x}_i^n) - \sum_p m_p \boldsymbol{x}_p^n \times (\boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{x}_p^n) \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p m_p \left(\sum_i w_{ip}^n \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{x}_i^n (\boldsymbol{x}_i^n)^T \right)^T : \boldsymbol{\epsilon} - \sum_p m_p \left(\boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{x}_p^n (\boldsymbol{x}_p^n)^T \right)^T : \boldsymbol{\epsilon} \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p m_p \left(\sum_i w_{ip}^n \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{x}_i^n (\boldsymbol{x}_i^n)^T - \boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{x}_p^n (\boldsymbol{x}_p^n)^T \right)^T : \boldsymbol{\epsilon} \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p m_p \left(\boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \left(\sum_i w_{ip}^n \boldsymbol{x}_i^n (\boldsymbol{x}_i^n)^T - \boldsymbol{x}_p^n (\boldsymbol{x}_p^n)^T \right) \right)^T : \boldsymbol{\epsilon} \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p m_p \left(\boldsymbol{B}_p^n (\boldsymbol{D}_p^n)^{-1} \boldsymbol{D}_p^n \right)^T : \boldsymbol{\epsilon} \\
&= \sum_p \boldsymbol{x}_p^n \times m_p \boldsymbol{v}_p^n + \sum_p m_p (\boldsymbol{B}_p^n)^T : \boldsymbol{\epsilon} \\
&= \boldsymbol{L}_{tot}^{P,n}
\end{aligned}$$

□

5.3.2 Grid to particle

Proposition 5.5. *Angular momentum is conserved during the APIC transfer from the grid to particles.* $\boldsymbol{L}_{tot}^{P,n+1} = \boldsymbol{L}_{tot}^{G,n+1}$ *under transfer 4.*

Proof. As before, the manipulation $(\boldsymbol{v}\boldsymbol{u}^T)^T : \boldsymbol{\epsilon} = \boldsymbol{u} \times \boldsymbol{v}$ is used to convert the permutation tensor into a cross product. The angular momentum on the particles after transferring from the grid is

$$\begin{aligned}
\boldsymbol{L}_{tot}^{P,n+1} &= \sum_p \boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} + \sum_p m_p (\boldsymbol{B}_p^{n+1})^T : \boldsymbol{\epsilon} \\
&= \sum_p \boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} + \sum_p m_p \left(\sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n)^T \right)^T : \boldsymbol{\epsilon} \\
&= \sum_p \boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} + \sum_p m_p \sum_i w_{ip}^n (\boldsymbol{x}_i^n - \boldsymbol{x}_p^n) \times \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_p \boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} - \sum_p m_p \sum_i w_{ip}^n \boldsymbol{x}_p^n \times \tilde{\boldsymbol{v}}_i^{n+1} + \sum_p m_p \sum_i w_{ip}^n \boldsymbol{x}_i^n \times \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_p \boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} - \sum_p m_p \boldsymbol{x}_p^n \times \sum_i w_{ip}^n \tilde{\boldsymbol{v}}_i^{n+1} + \sum_i \left(\sum_p m_p w_{ip}^n \right) \boldsymbol{x}_i^n \times \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_p \boldsymbol{x}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} - \sum_p m_p \boldsymbol{x}_p^n \times \boldsymbol{v}_p^{n+1} + \sum_i m_i^n \boldsymbol{x}_i^n \times \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_p (\boldsymbol{x}_p^{n+1} - \boldsymbol{x}_p^n) \times m_p \boldsymbol{v}_p^{n+1} + \sum_i \boldsymbol{x}_i^n \times m_i^n \tilde{\boldsymbol{v}}_i^{n+1} \\
&= \sum_p \Delta t \boldsymbol{v}_p^{n+1} \times m_p \boldsymbol{v}_p^{n+1} + \boldsymbol{L}_{tot}^{G,n+1} \\
&= \boldsymbol{L}_{tot}^{G,n+1}
\end{aligned}$$

□