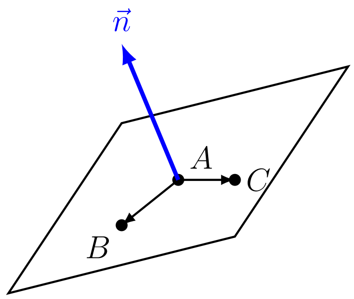
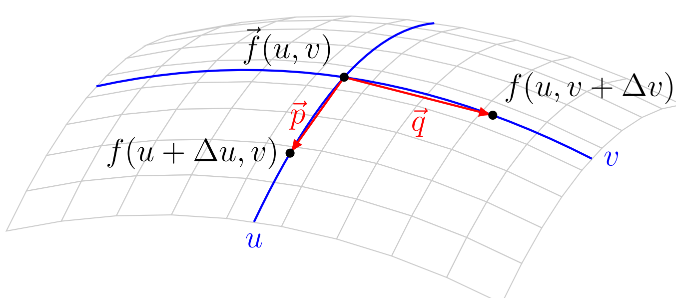
Calculating Surface Normals

1 Normals of a Plane



For a plane defined by three points A, B, and C, the normal vector $\vec{n}$ is found using the cross product of two vectors lying on the plane. We can get a vector orthogonal to the vectors connecting the points using a cross product, so $\vec{N} = (B - A) \times (C - A)$. Then, we just need to normalize it to get a direction $\vec{n} = \frac{\vec{N}}{\|\vec{N}\|}$.

2 Parametric Surfaces



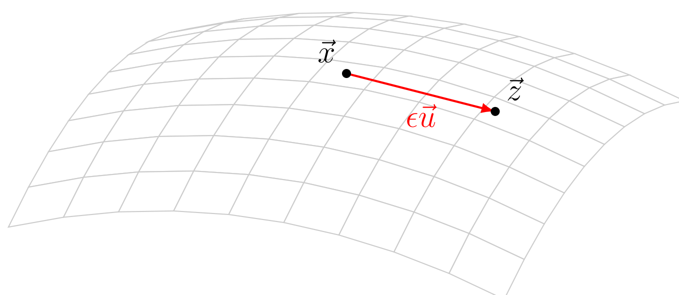
A parametric surface is defined by a function $\vec{f}(u, v)$. To find the normal, we calculate the tangent vectors by taking partial derivatives with respect to u and v.

$$N = \vec{p} \times \vec{q} = (f(u + \Delta u, v) - \vec{f}(u, v)) \times (f(u, v + \Delta v) - \vec{f}(u, v)) \to 0$$

$$\frac{N}{\Delta u \Delta v} = \left(\frac{f(u + \Delta u, v) - \vec{f}(u, v)}{\Delta u}\right) \times \left(\frac{f(u, v + \Delta v) - \vec{f}(u, v)}{\Delta v}\right) \to \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$$

$$n = \frac{\vec{N}}{\|\vec{N}\|} = \frac{\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}}{\|\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}\|}$$

3 Implicit Surfaces



An implicit surface is defined as the set of points where $f(\vec{x}) = 0$. We have already seen some of these. Plane: $(\vec{x} - \vec{p}) \cdot \vec{n} = 0$. Sphere: $\|\vec{x} - \vec{c}\|^2 - r^2 = 0$. In this case, we are given a point $\vec{x}$ on the surface, and we want to compute the normal at that location. Unlike the parametric case, we cannot generate new points on the surface. But we can consider what would happen if we were given a point $\vec{z} = \vec{x} + \epsilon\vec{u}$ nearby on the surface. Then,

$$f(\vec{x}) = f(\vec{z}) = 0$$

$$0 = \frac{f(\vec{z}) - f(\vec{x})}{\epsilon} = \frac{f(\vec{x} + \epsilon\vec{u}) - f(\vec{x})}{\epsilon}$$

By imagining that we are given points increasingly close, we can consider the limit $\epsilon \to 0$. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{x} = \langle x_1, x_2, x_3 \rangle$. Then,

$$\begin{aligned} 0 &= \lim_{\epsilon \to 0} \frac{f(\vec{x} + \epsilon\vec{u}) - f(\vec{x})}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\frac{d}{d\epsilon} f(\vec{x} + \epsilon\vec{u})}{1} \qquad \text{L'Hôpital} \\ &= \lim_{\epsilon \to 0} \frac{d}{d\epsilon} f(\vec{x} + \epsilon\vec{u}) \end{aligned}$$

Looking at the components,

$$\begin{aligned} \frac{d}{d\epsilon} f(\vec{x} + \epsilon\vec{u}) &= \frac{d}{d\epsilon} f(x_1 + \epsilon u_1, x_2 + \epsilon u_2, x_3 + \epsilon u_3) \\ &= \frac{\partial f}{\partial x_1} u_1 + \frac{\partial f}{\partial x_2} u_2 + \frac{\partial f}{\partial x_3} u_3 = \nabla f \cdot \vec{u} \end{aligned}$$

Here, ∇f is the gradient:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix}$$

This tells us that ∇f is orthogonal to $\vec{u}$, which is an arbitrary vector in the tangential direction. Since ∇f is orthogonal to tangential directions, it must point in the normal direction. Thus,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|}$$