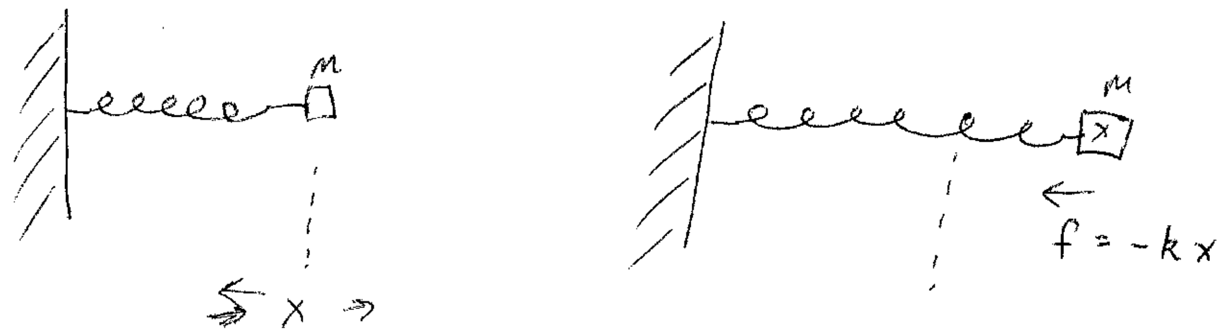


$x > 0$
$f < 0$

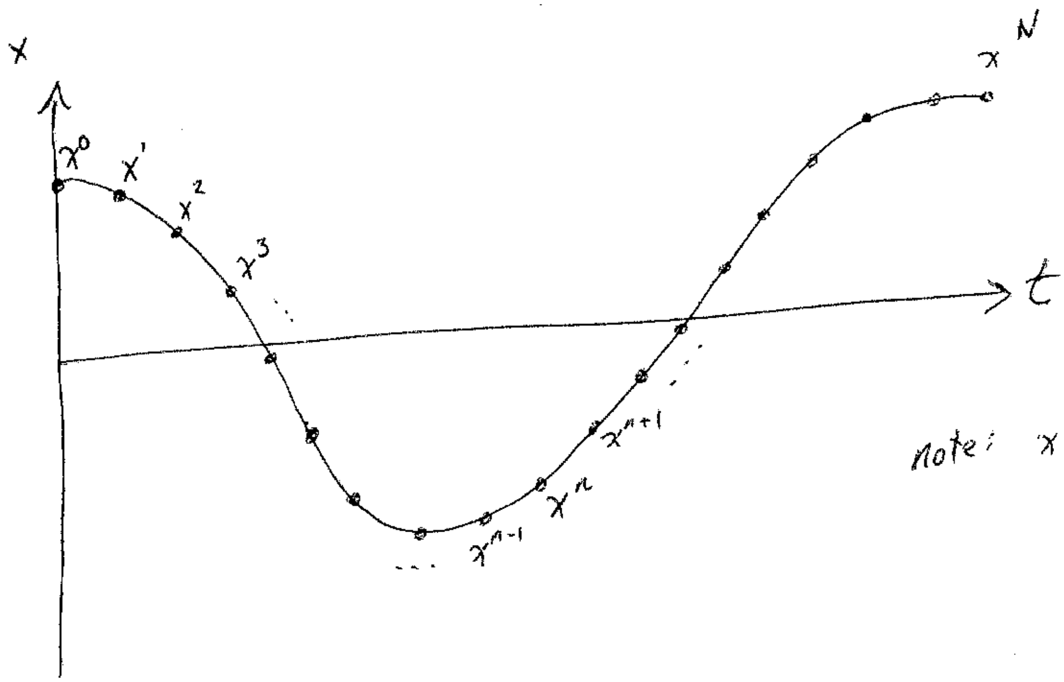
$$V = \frac{dx}{dt} = \dot{x}$$

$$a = \dot{V} = \ddot{x}$$

$$f = ma = m\ddot{x}$$

$$\boxed{m\ddot{x} = -kx}$$ ODE second order

$$\left.\begin{aligned} \dot{x} &= V \\ m\dot{V} &= -kx \\ \dot{V} &= -\frac{k}{m}x \end{aligned}\right\}$$ system first order



N steps

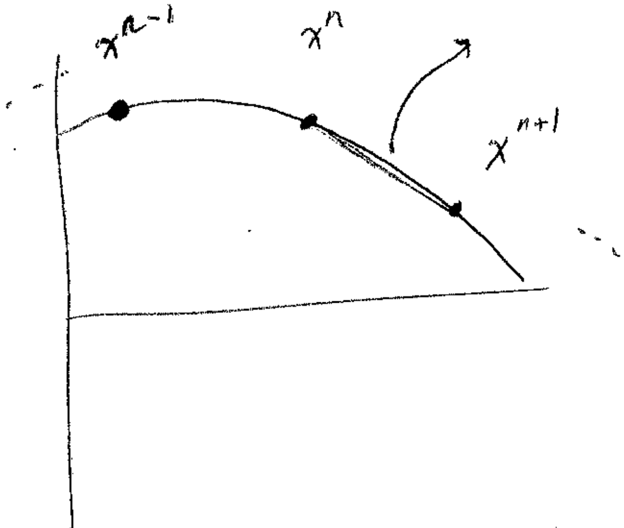
T final time

$$\Delta t = \frac{T}{N}$$

note: x^n is not power

* derivative in time

$$\frac{x^{n+1}-x^n}{\Delta t} = \frac{x(t^{n}+\Delta t)-x(t^n)}{\Delta t} \approx \dot{x}(t^{n+1})+O(\Delta t)$$
$$\approx \dot{x}(t^n)+O(\Delta t)$$



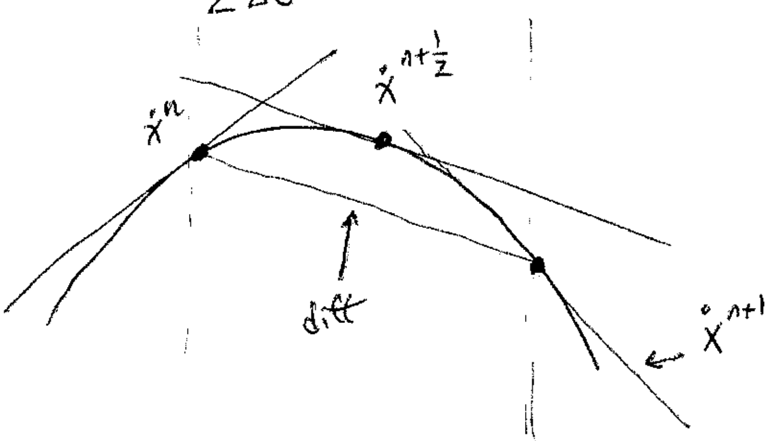
$$\frac{x^{n+1}-x^n}{\Delta t}=v^n \qquad \frac{v^{n+1}-v^n}{\Delta t} = -\frac{k}{m}x^n$$ Forward Euler

$$\frac{x^{n+1}-x^n}{\Delta t}=v^{n+1} \qquad \frac{v^{n+1}-v^n}{\Delta t} = -\frac{k}{m}x^{n+1}$$ Backward Euler

→ system of equations

→ in general, nonlinear!

$$x(t^n+\Delta t) \approx x(t^n)+\Delta t\,\dot{x}(t^n)+\frac{\Delta t^2}{2}\ddot{x}(t^n)+\frac{\Delta t^3}{6}\dddot{x}(t^n)+\cdots$$
$$\frac{x(t^n+\Delta t)-x(t^n)}{\Delta t} \approx \dot{x}(t^n)+\frac{\Delta t}{2}\ddot{x}(t^n)+\frac{\Delta t^2}{6}\dddot{x}(t^n)+\cdots$$ forward diff
$$x(t^n-\Delta t) \approx x(t^n)-\Delta t\,\dot{x}(t^n)+\frac{\Delta t^2}{2}\ddot{x}(t^n)-\frac{\Delta t^3}{6}\dddot{x}(t^n)+\cdots$$
$$\frac{x(t^n+\Delta t)-x(t^n-\Delta t)}{2\Delta t} \approx \dot{x}(t^n)+\frac{\Delta t^2}{6}\dddot{x}(t^n)+\cdots = \dot{x}(t^n)+O(\Delta t^2)$$ central diff



$$\frac{x^{n+1} - x^n}{\Delta t} = \frac{v^{n+1} + v^n}{2} = \underset{\uparrow \text{ def.}}{v^{n+\frac{1}{2}}}$$

$$\frac{v^{n+1} - v^n}{\Delta t} = -\frac{k}{2}\, \underset{\uparrow \text{ def.}}{x^{n+\frac{1}{2}}} = -\frac{k}{2}\left(\frac{x^{n+1} + x^n}{2}\right)$$

→ much more accurate

→ system of equations.

in general: $\dot{x} = f(x)$

$$\left.\begin{aligned} &\frac{x^{n+1} - x^n}{\Delta t} = f(x^n) && \text{Forward Euler} \\ &\frac{x^{n+1} - x^n}{\Delta t} = f(x^{n+1}) && \text{Backward Euler} \end{aligned}\right\} \ O(\Delta t) \text{ global error}$$

$$\left.\begin{aligned} &\frac{x^{n+1} - x^n}{\Delta t} = f(x^{n+\frac{1}{2}}) = f\left(\frac{x^{n+1} + x^n}{2}\right) && \text{midpoint rule} \\ &\frac{x^{n+1} - x^n}{\Delta t} = \frac{f(x^{n+1}) + f(x^n)}{\Delta t} && \text{trapezoid rule} \end{aligned}\right\} \ O(\Delta t^2) \text{ global error}$$

$\boxed{\text{stability}}$ $\dot{x} = \lambda x \Rightarrow x = e^{\lambda t} x^0$ decays if $\lambda < 0$

$$\frac{x^{n+1} - x^n}{\Delta t} = \lambda x^n \Rightarrow x^{n+1} = x^n + \Delta t \, \lambda x^n \Rightarrow x^{n+1} = (1 + \Delta t \, \lambda) x^{n+1}$$

$$= (1 + \Delta t \lambda)^{n+1} x^0$$

decays if $|1 + \Delta t \lambda| < 1$

$$-1 \leq 1 + \Delta t \lambda < 1$$

$$\underset{+}{-2} < \underset{-}{\Delta t \lambda} < \underset{\text{trivial}}{0} \Rightarrow \Delta t < \frac{-2}{\lambda}$$

time step restriction!

otherwise exp. growth. ⇒ unstable

$$\frac{x^{n+1} - x^n}{\Delta t} = \lambda x^{n+1} \qquad (1 - \lambda \Delta t) x^{n+1} = x^n \qquad x^{n+1} = \frac{1}{(1 - \lambda \Delta t)^{n+1}} x^0$$

$|1 - \lambda \Delta t| > 1$ always true if $\lambda < 0$ $\Delta t > 0$ unconditionally stable!

FE → cheap, simple

BE → stable, larger steps