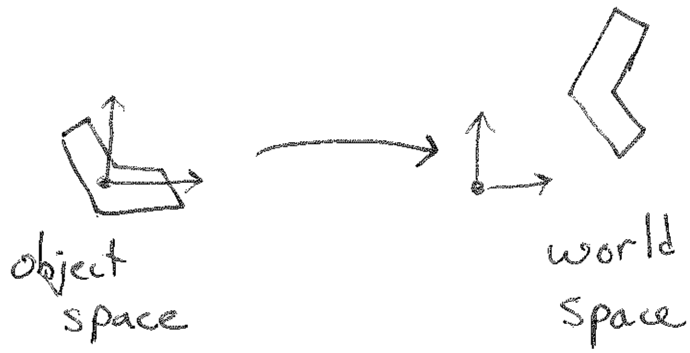
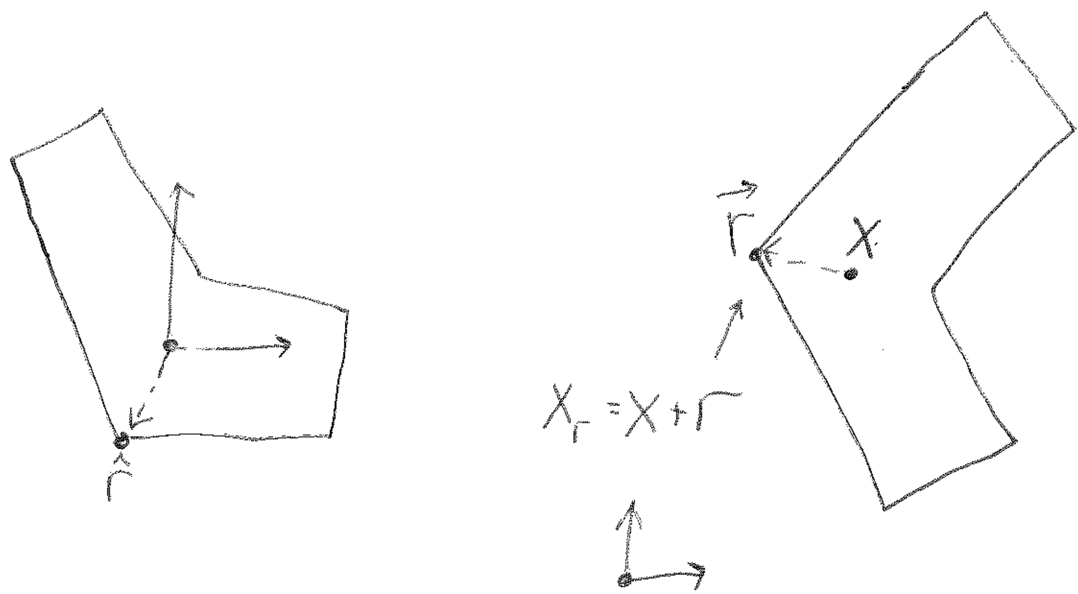
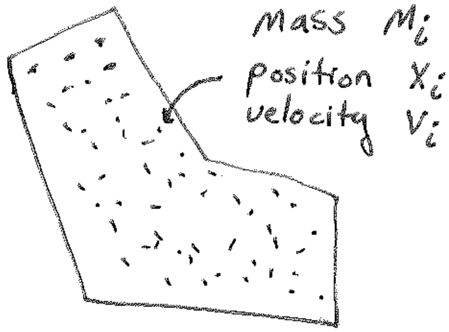
* Rigid Bodies

→ no shape change.





$$x_r = x + r = x + R\hat{r} \qquad r = R\hat{r}$$

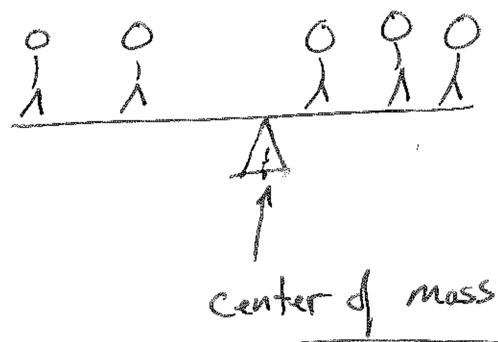


$$\text{mass} \quad m = \sum_i m_i$$

$$\text{center of mass} = x_c = \frac{1}{m} \sum_i m_i x_i$$

choose: $x = x_c$

$$x = \frac{1}{m} \sum_i m_i x_i$$

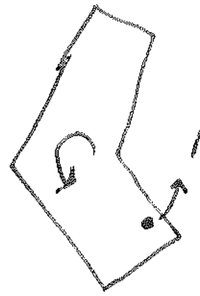


$$x = \frac{1}{m} \sum_i m_i x_i = \frac{1}{m} \sum_i m_i \left(x + R\hat{r}_i \right)$$

$$= \frac{1}{m} \left(\sum_i m_i \right) x + \frac{1}{m} \sum_i m_i R \hat{r}_i$$

$$= x + \frac{1}{m} R \left(\sum_i m_i \hat{r}_i \right)$$

$$\Rightarrow \sum_i m_i \hat{r}_i = 0$$

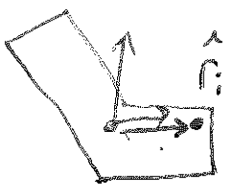


$M_i \quad X_i \quad V_i$

world space

$$V_i = \dot{X}_i = \left(X + R\hat{r}_i\right)' = \dot{X} + \dot{R}\hat{r}_i = V + \dot{R}\hat{r}_i = V + \underline{\dot{R}R^T r_i}$$

$$V = \dot{X}$$



$\hat{r}_i$

$$r = R\hat{r}$$

$$\hat{r} = R^{-1} r = R^T r$$

$$\underset{\substack{\uparrow \\ \text{identity}}}{RR^T = \delta} \quad \Rightarrow \dot{R}R^T + R\dot{R}^T = 0$$

$$= (\dot{R}R^T) + (\dot{R}R^T)^T$$

if $A + A^T = 0 \Rightarrow A$ is skew symmetric

$$\dot{R}R^T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \omega^* \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$\omega^* u = \omega \times u$$

$$V_i = V + \dot{R}R^T r_i = V + \omega^* r_i = V + \underset{\substack{\text{angular} \\ \text{velocity}}}{\omega} \times r_i$$

$\|\omega\| \Rightarrow$ rotation rate

$\frac{\omega}{\|\omega\|} \Rightarrow$ axis of rotation

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\dot{R}(\theta) = \begin{pmatrix} -\sin\theta\,\dot{\theta} & -\cos\theta\,\dot{\theta} \\ \cos\theta\,\dot{\theta} & -\sin\theta\,\dot{\theta} \end{pmatrix}$$

$$\omega^* = \dot{R}R^T = \dot{\theta}\begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \dot{\theta}\left(\begin{array}{c|c} -sc+cs & -s^2-c^2 \\ \hline c^2+s^2 & cs-sc \end{array}\right)$$

$$= \dot{\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\omega = \left(\pm\dot{\theta}\right)$$

$$\|\omega\| = |\dot{\theta}|$$

Momentum $\rightarrow$ $p = MV$

$\uparrow$
momentum

$$p = \sum_i m_i v_i = \sum_i m_i (V + \omega \times r_i)$$

$$= \left(\sum_i m_i\right) V + \sum_i m_i \, \omega \times r_i$$

$$= MV + \omega \times \sum_i m_i r_i$$

$$= MV + \omega \times \left(R \underbrace{\sum_i m_i \hat{r}_i}_{0}\right)$$

$$\underline{= MV}$$