x

$$m_1 \ddot{x}_1 = f_1$$
$$m_2 \ddot{x}_2 = f_2$$
$$m_3 \ddot{x}_3 = f_3$$
$$\vdots$$
$$m_n \ddot{x}_n = f_n$$

$$M\ddot{x} = f$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad M \begin{pmatrix} m_1 & & & & & & & & & \\ & m_1 & & & & & & & & \\ & & m_1 & & & & & & & \\ & & & m_2 & & & & & & \\ & & & & m_2 & & & & & \\ & & & & & m_2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & m_n & & \\ & & & & & & & & m_n & \\ & & & & & & & & & m_n \end{pmatrix}$$

$\vec{f}_1 \to$ 3-vec

$\vec{f} \to$ 3n-vec

$$\dot{x} = v \qquad M\dot{v} = f$$

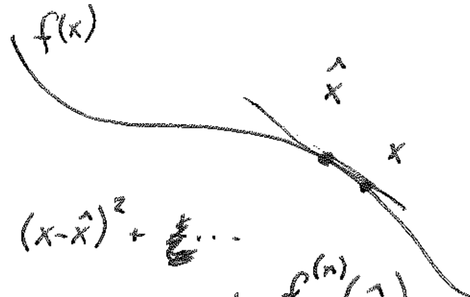
Forward Euler: $\dfrac{x^{n+1} - x^n}{\Delta t} = v^n \qquad M\dfrac{v^{n+1} - v^n}{\Delta t} = f^n = f(x^n)$

Backward Euler: $\dfrac{x^{n+1} - x^n}{\Delta t} = v^{n+1} \qquad M\dfrac{v^{n+1} - v^n}{\Delta t} = f^{n+1} = f(x^{n+1})$

Trapezoid Rule: $\dfrac{x^{n+1} - x^n}{\Delta t} = \dfrac{v^n + v^{n+1}}{2} \qquad M\dfrac{v^{n+1} - v^n}{\Delta t} = \dfrac{f(x^n) + f(x^{n+1})}{2}$

Midpoint Rule: $\dfrac{x^{n+1} - x^n}{\Delta t} = \dfrac{v^n + v^{n+1}}{2} \qquad M\dfrac{v^{n+1} - v^n}{\Delta t} = f\left(\dfrac{x^n + x^{n+1}}{2}\right)$

Taylor Series



$$f(x) = f(\hat{x}) + (x-\hat{x})f'(\hat{x}) + \frac{1}{2} f''(\hat{x})\,(x-\hat{x})^2 + \dots + \frac{1}{n!} f^{(n)}(\hat{x})\,(x-\hat{x})^n$$

$$\vec{f}(\vec{x}) = \vec{f}(\vec{x}_0) + \left(\frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}_0)\right)(\vec{x}-\vec{x}_0)$$

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \frac{\partial \vec{f}}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

scalars

$$\boxed{\nabla \vec{f}}$$

$$M\,\frac{v^{n+1} - v^n}{\Delta t} = f(x^{n+1}) \qquad \frac{x^{n+1} - x^n}{\Delta t} = v^{n+1} \qquad x^{n+1} = x^n + \Delta t\, v^{n+1}$$

$$M\,\frac{v^{n+1} - v^n}{\Delta t} = f(x^n + \Delta t\, v^{n+1})$$

assume guess $\hat{v}$

$\hat{v} \approx v^{n+1}$

$$v^{n+1} - \hat{v} = \Delta v$$

$$M\,\frac{\Delta v + \hat{v} - v^n}{\Delta t} = f(\underbrace{x^n + \Delta t\hat{v}}_{\hat{x}} + \Delta t\,\Delta v) = f(\hat{x} + \Delta t\,\Delta v) \approx f(\hat{x}) + \left(\frac{\partial f}{\partial x}(\hat{x})\right)\Delta t\,\Delta v$$

$$M\,\Delta v - \Delta t^2\,\frac{\partial f}{\partial x}(\hat{x})\,\Delta v = M(v^n - \hat{v}) + \Delta t\, f(\hat{x})$$

$$\left(M - \Delta t^2 \frac{\partial f}{\partial x}(\hat{x})\right)\Delta v = M\left(v^n - \hat{v}\right) + \Delta t\, f(\hat{x})$$

Solve linear system $\rightarrow \Delta v$

old estimate: $\hat{v}$

new estimate: $\hat{v} + \Delta v$

can be repeated, converges to v^{n+1}.

Newton's method

Raphson

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minimize $g(\vec{x})$ $\qquad \underline{\frac{\partial g}{\partial x} = 0}$

$\uparrow$

nonlinear system of equations