Math Review

CS 130

1. Points
   - Locations: P, Q, R
2. Vectors
   - Direction, magnitude
   - No location
   - Used to indicate quantities like change in position, velocity, force, etc.
   - $\mathbf{u}, \mathbf{v}, \mathbf{w}$ or $\vec{u}, \vec{v}, \vec{w}$
   - Note that positions are very often represented with vectors (displacement from the origin).
   - Vector operations

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \qquad k \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

   - Coordinates; a_1, a_2, a_3 are coordinates, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are a basis

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

   - In practice, the basis is normally understood (usually the standard one above), and only the coordinates are written explicitly.
3. Linear operator
   - Special type of function $f(x)$
   - Defining property: $af(x) + bf(y) = f(ax + by)$
   - Typical form $f(x) = ax$, though not always. (Differentiation is a linear operator.)
   - For example, if x, y are vectors, then the linear operators are matrices. That is, $f(\mathbf{u}) = \mathbf{Mu}$.
   - Note that $f(0) = 0$.
4. Affine operator
   - Typical form: $f(x) = ax + b$.
   - Note that $f(0)$ is not necessarily 0.
   - The definition is fairly general. x and b could be vectors; a could be a matrix.
5. Lines
   - $L(t) = P + t\mathbf{u}$.

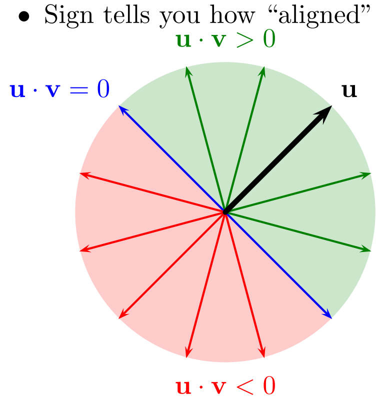
- Point on the line: P
- Direction of the line: $\mathbf{u}$
- t is a parameter that tells where a point is along the line.

6. Line segment

   - Straight line connecting endpoints P, Q.
   - $L(t) = (1-t)P + tQ$; $t \in [0,1]$.
   - Note: $L(t) = (1-t)P + tQ = P + t(Q-P)$, $\mathbf{u} = Q - P$.

7. Dot product

   - $\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$
   - Well defined for any size of vector (any number of components), but both vectors must have the same size.
   - $\mathbf{u} \cdot \mathbf{u} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2 = \|\mathbf{u}\|^2$; $\|\mathbf{u}\|$ is the length of the vector.
   - $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$; θ is the angle between the vectors.
   - Sign tells you how "aligned" two vectors are:



8. Cross Product

   - Defined in 3D only.

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} &= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \\
&= \underbrace{\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}_{\text{(notation abuse)}} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}
\end{aligned}
$$

   - Note that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is a vector.
   - Length: $\|\mathbf{w}\| = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$.

- Direction: $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$.
- Note that both $\mathbf{w}$ and $-\mathbf{w}$ have these properties; need a convention to disambiguate direction.
- How I remember it: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.
- Others use "right hand rule" and similar devices.
- Note that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$, since $\theta = 0$ and $\|\mathbf{u} \times \mathbf{u}\| = \|\mathbf{u}\|\|\mathbf{u}\| \sin\theta = \mathbf{0}$

9. Dot product and cross product have lots of useful properties, which are straightforward to check:

$$\begin{array}{ccc} \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} & (a\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v}) & (\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} \\ \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} & (a\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (a\mathbf{v}) = a(\mathbf{u} \times \mathbf{v}) & (\mathbf{u} + \mathbf{w}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v} + \mathbf{w} \times \mathbf{v} \\ (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} & \|\mathbf{u}\|^2\|\mathbf{v}\|^2 = (\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2 & \end{array}$$

10. Matrices

- Table of numbers

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}$$

- Entries are indexed a_{ij}; row is i, column is j.
- Matrices have dimensions that indicate their size and shape: $\mathbf{M}$ is 3×3, $\mathbf{N}$ is 4×2.
- Matrix-vector multiply:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \\ a_{31}u_1 + a_{32}u_2 + a_{33}u_3 \end{pmatrix}$$

- Pattern: $\mathbf{v} = \mathbf{M}\mathbf{u}$ is calculated as $v_i = \sum_j m_{ij} u_j$
- Can be thought of as a linear operator: $\mathbf{v} = f(\mathbf{u}) = \mathbf{M}\mathbf{u}$.
- Composition: $f(\mathbf{u}) = \mathbf{M}\mathbf{u}$, $g(\mathbf{u}) = \mathbf{P}\mathbf{u}$, $h(\mathbf{u}) = f(g(\mathbf{u}))$ or $h = f \circ g$.

$$\mathbf{w} = f(g(\mathbf{u})) = f(\mathbf{P}\mathbf{u}) = \mathbf{M}(\mathbf{P}\mathbf{u}) = (\mathbf{M}\mathbf{P})\mathbf{u} = \mathbf{Q}\mathbf{u}$$

$$w_i = \sum_j m_{ij} \left(\sum_k p_{jk} u_k \right) = \sum_k \underbrace{\left(\sum_j m_{ij} p_{jk} \right)}_{q_{ik}} u_k$$

$$\mathbf{Q} = \mathbf{M}\mathbf{P} \Leftrightarrow q_{ik} = \sum_j m_{ij} p_{jk}$$

- This is the rule for matrix multiplication.
- Transpose: $\mathbf{N} = \mathbf{M}^T$ is defined by $n_{ij} = m_{ji}$