CS141: Intermediate Data Structures and Algorithms

NP-Completeness: A Brief Summary

Amr Magdy

Why Studying NP-Completeness?

- Two reasons:
  1. In almost all cases, if we can show a problem to be **NP-complete** or **NP-hard**, the best we can achieve (NOW) is mostly exponential algorithms.
     - This means we cannot solve large problem sizes efficiently
  2. If we can solve only one NP-complete problem efficiently, we can solve ALL NP problems efficiently (major breakthrough)
- More details come on what does these mean

Topic Outline

1. Decision Problems
2. Complexity Classes
   - P
   - NP
     - Polynomial Verification
     - Examples
3. NP-hardness
   - Polynomial Reductions
4. NP-Complete Problems
   - Definition and Examples

Decision Problems

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Decision Problems

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  - Examples:
    - Is graph G connected?
    - Is there a path P:u→v of cost 100?
    - Is there a common subsequence of strings A and B of length 5?
    - …etc

Computation

- Decision Problems

Decidable Problems: problems that can be decided using a Turing machine (our computer)	Undecidable Problems: problems that cannot be decided using a Turing machine. Example: the halting problem

Computation

- Decision Problems are classified into different complexity classes based on time complexity and space complexity

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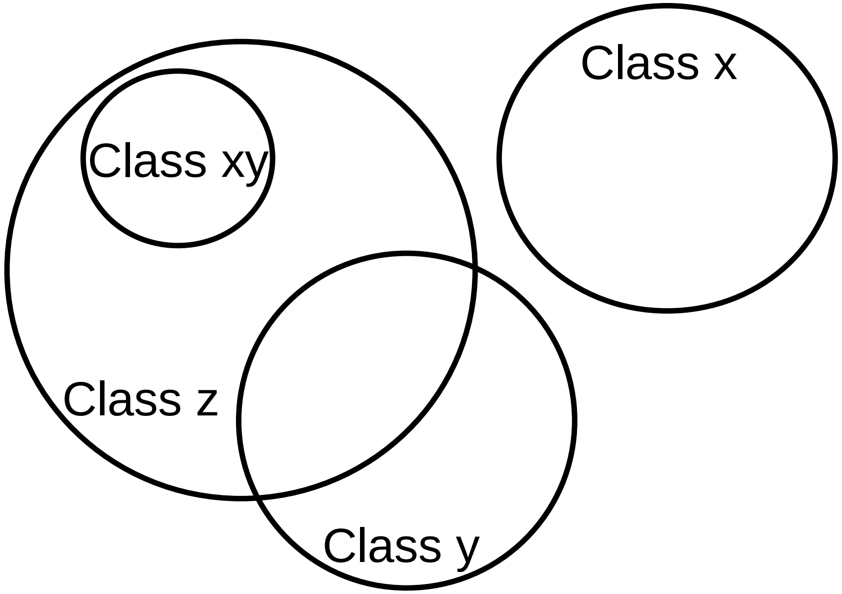

Complexity classes

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  - Either in time complexity
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- In this course, our discussion is limited to only three time complexity classes: P, NP, and NP-hard
  - Other courses cover more content (e.g., Theory of Computation course)

P

- P is a complexity class of problems that are ***decidable*** (for simplicity ***solvable***) in **polynomial-time**, i.e., $O(n^k)$
  - where <u>n the input length</u> and <u>k is constant</u>

P

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  - where n the input length and k is constant
- Examples:
  - Shortest paths in graph
  - Matrix chain multiplication
  - Activity scheduling problem
  - ….

NP

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- For simplicity, given a solution of an NP problem, we can verify in polynomial time $O(n^k)$ if this solution is correct

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- Examples:
  - Is bipartite graph? Given two subsets of nodes, verify it is bipartite
  - Max clique: Given a clique and k, verify it is actually a clique of size k
  - Shortest path: Given a path of cost C, verify it is a path and of cost C
  - …..

Is P $\subset$ NP?

- Yes
- What does this mean?
  - Every problem that is solvable in polynomial time is verifiable in polynomial time as well

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- What does this mean?

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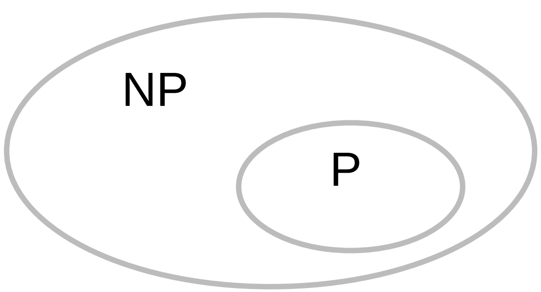

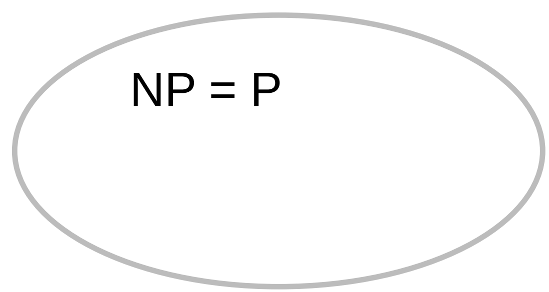

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We could solve the biggest problem in maths in the next decade

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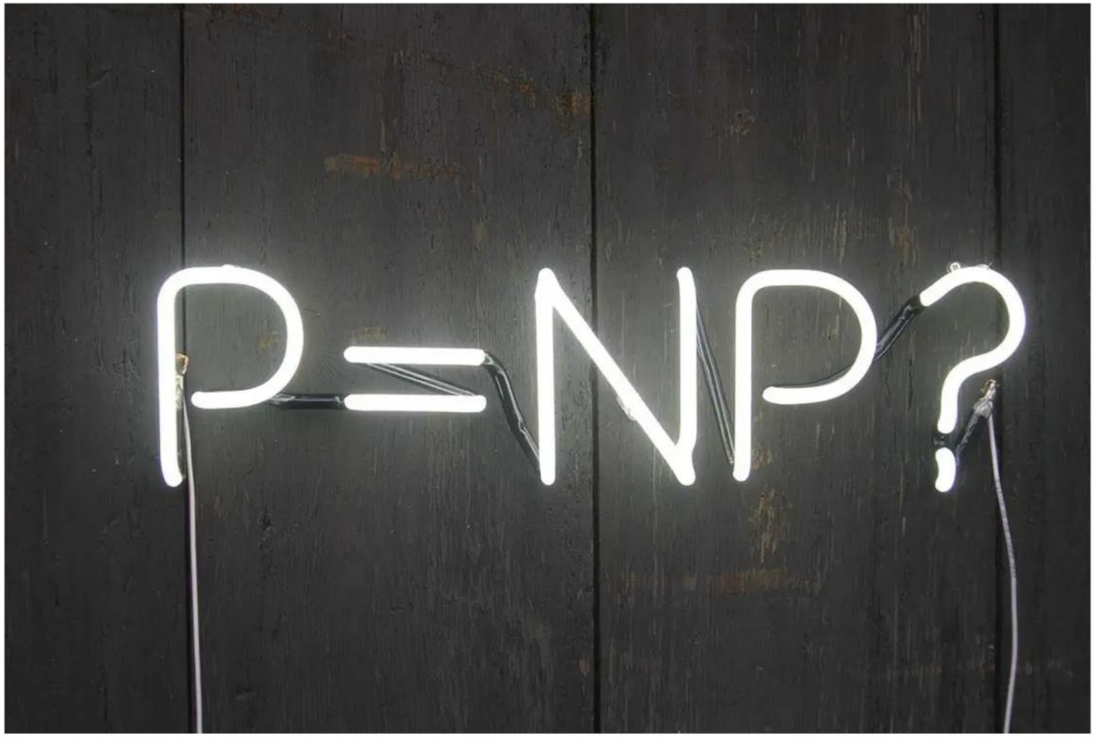


P is not NP? That is the question

By **Jacob Aron**

One of the biggest open problems in mathematics may be solved within the next decade, according to a poll of computer scientists. A solution to the so-called P versus NP problem is worth $1 million and could have a profound effect on computing, and perhaps even the entire world.

NP-hard Problems

- Informally:

an NP-hard problem B is a problem that is at least as hard as the hardest problems in NP class

- Formally:

B is NP-hard if $\forall$ A $\in$ NP, A $\leq_P$ B
(i.e., A is polynomially reducible to B)

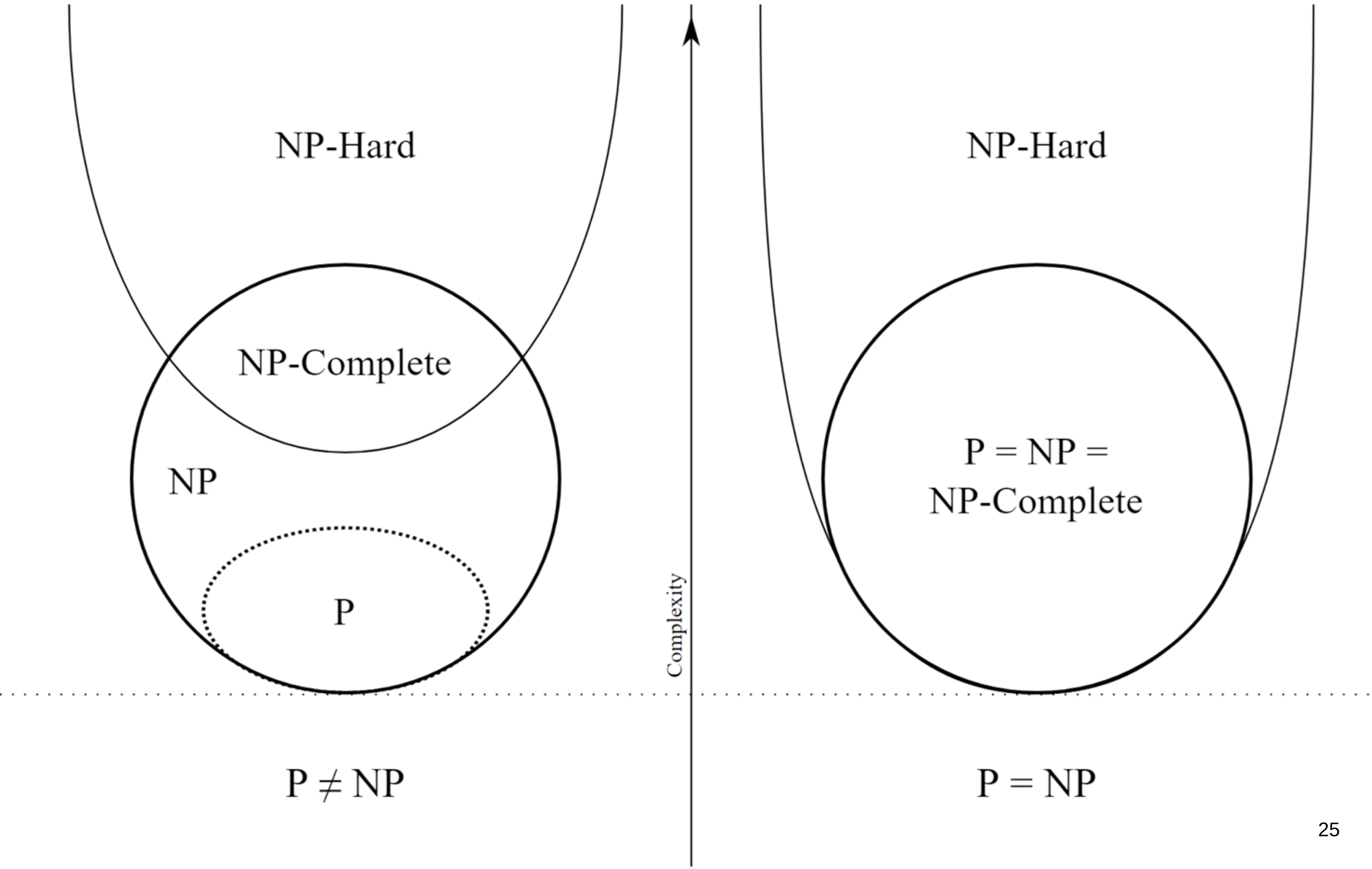
Polynomial Reductions

- Polynomial reduction A $\leq_P$ B is converting an instance of A into an instance of B in polynomial time.

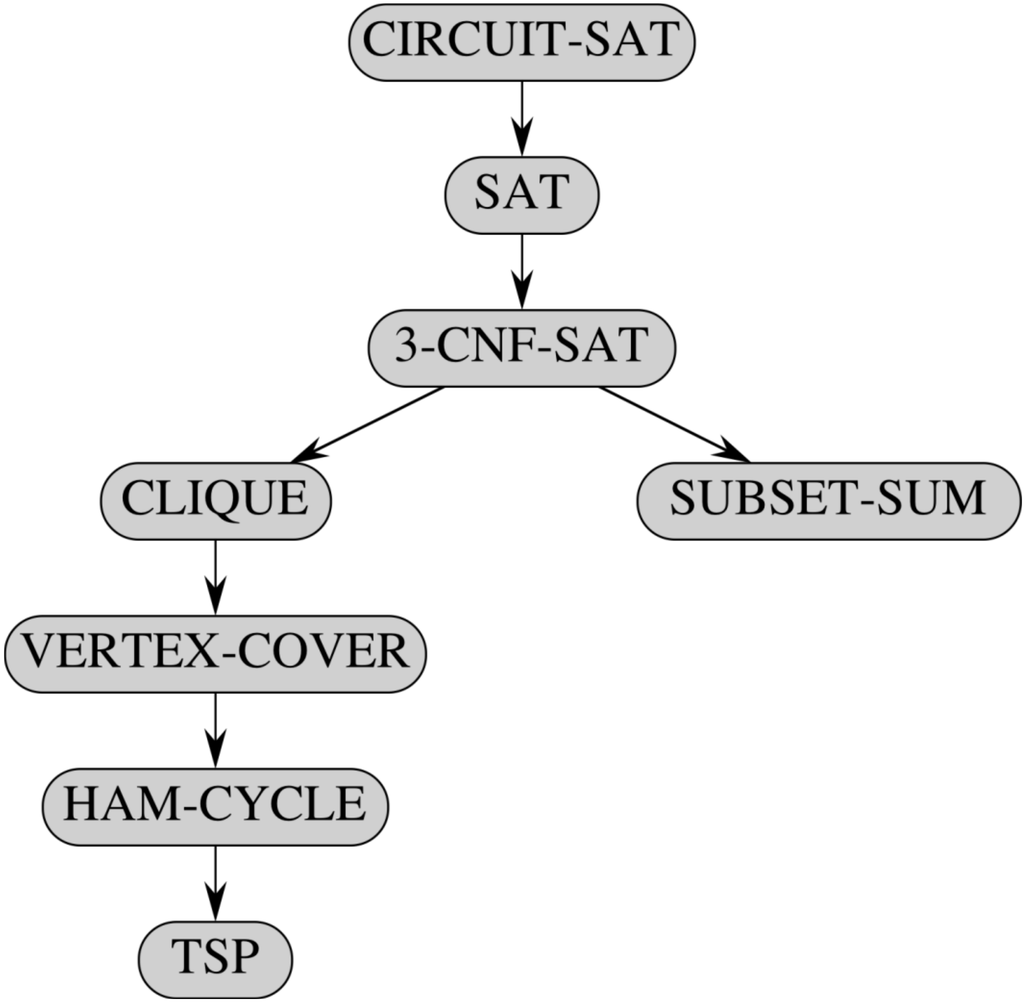
NP-Complete Problems

- B is NP-complete problem if:
  1. B ∈ NP
  2. B is NP-hard

NP-Complete Problems

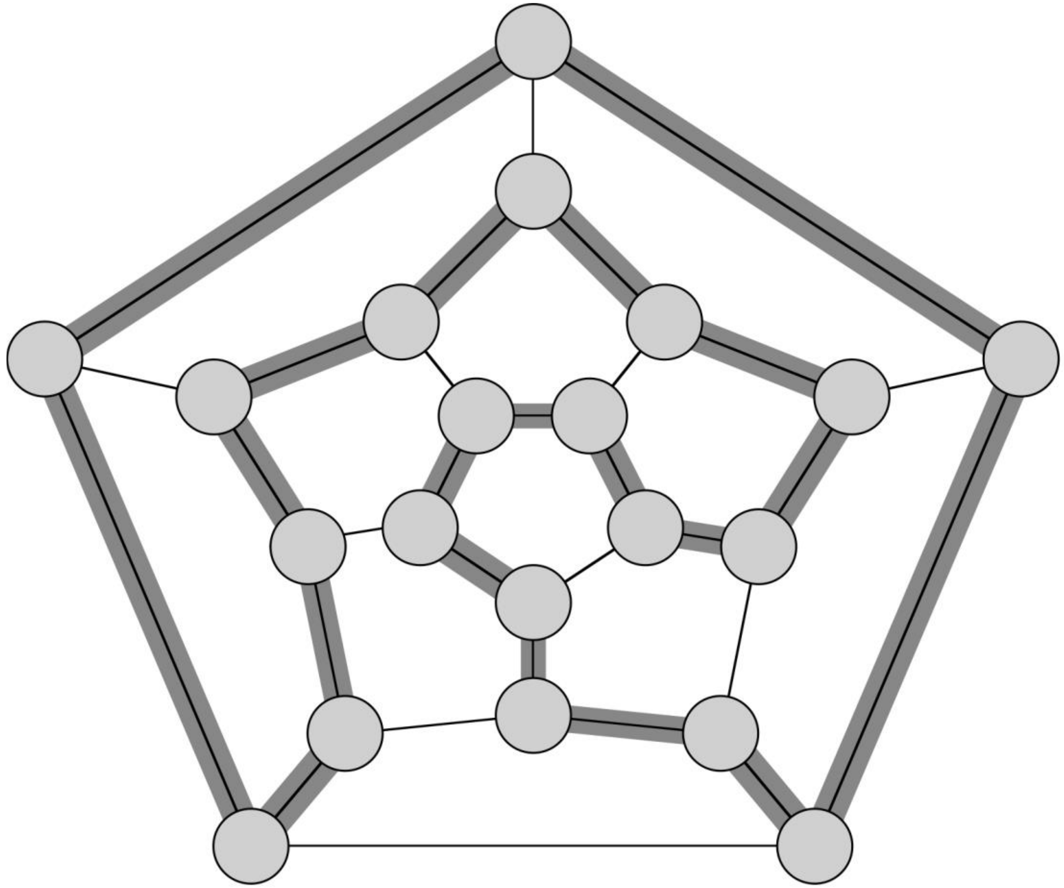


NP-Complete Problems: Examples



NP-Complete Problems: Examples

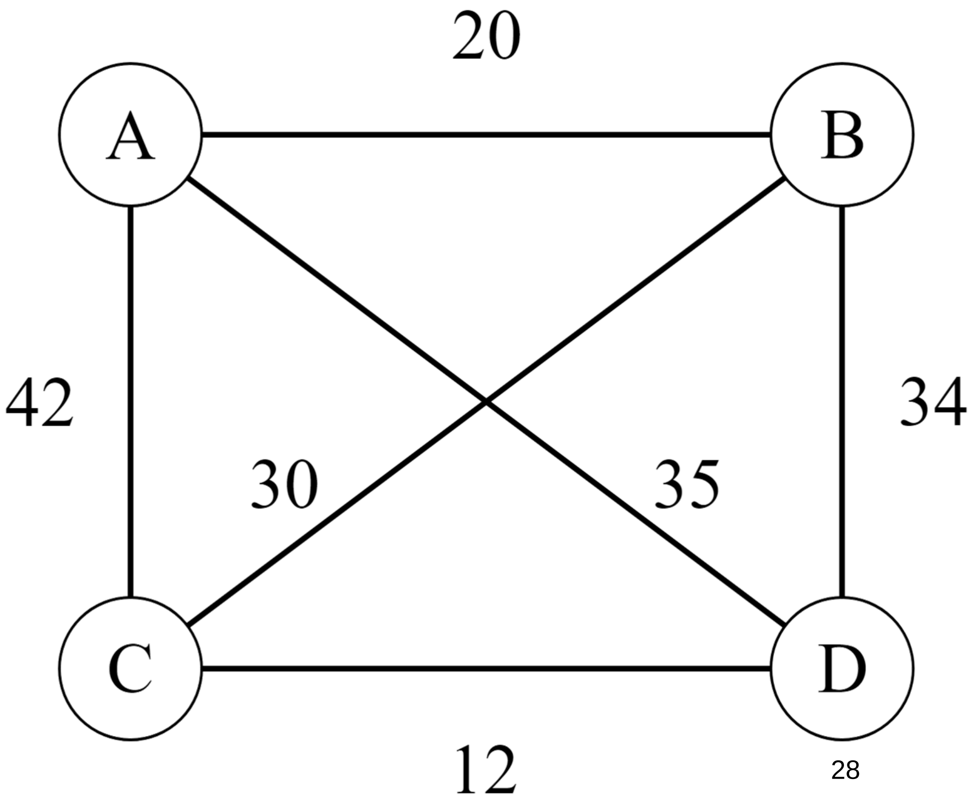
- **Hamiltonian Cycle Problem**: Given an undirected or directed graph G, is there a cycle in G that visits each vertex exactly once?



NP-Complete Problems: Examples

- Example: **Travelling Salesman Problem**

  Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?



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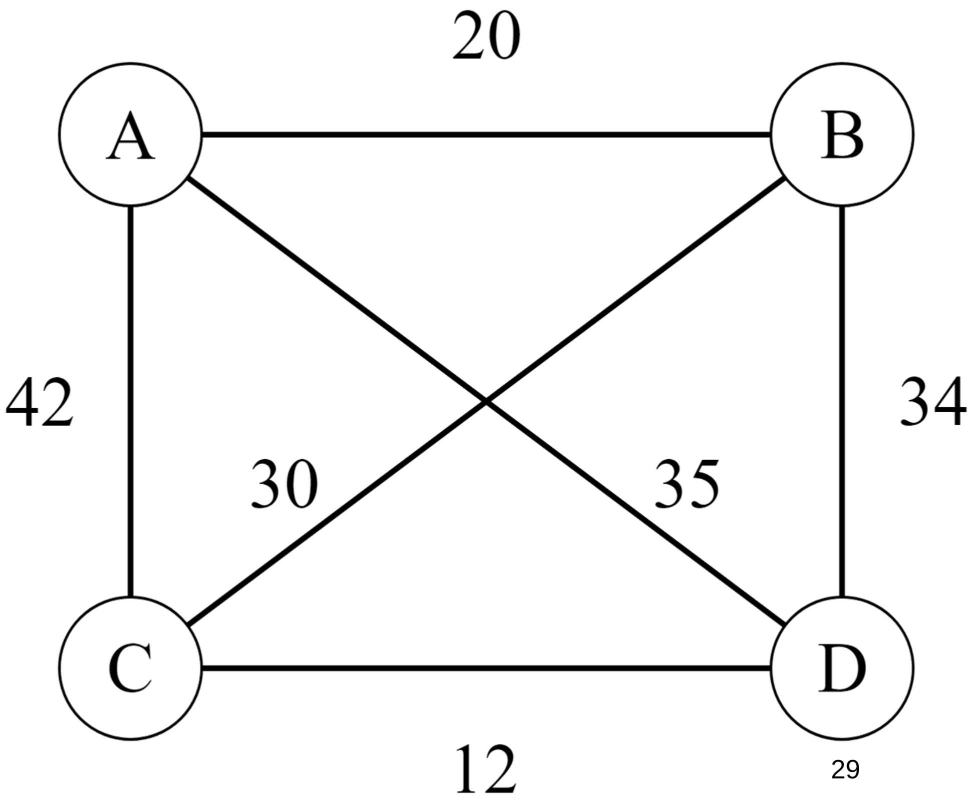

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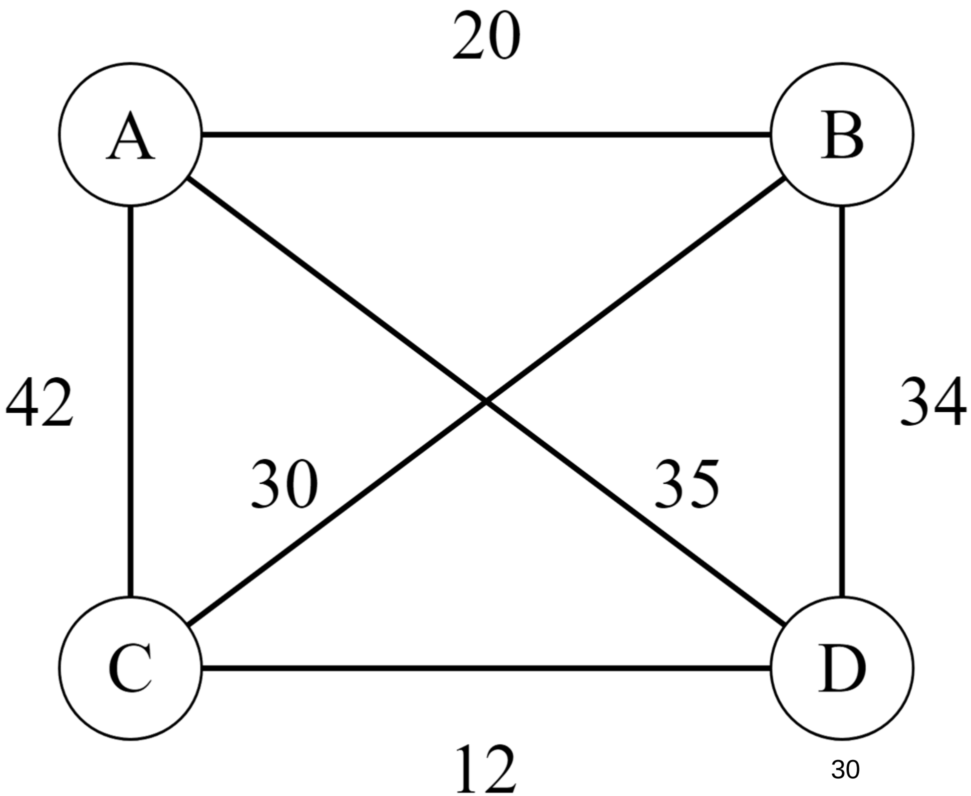

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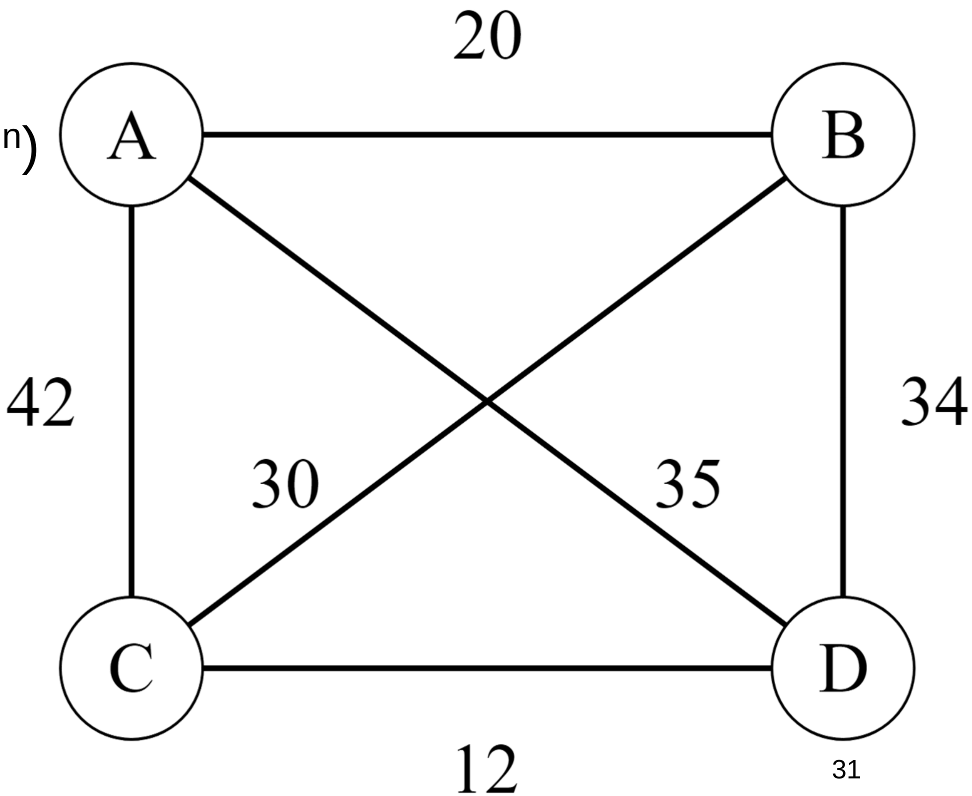
  - Dynamic programming: O($n2^n$)



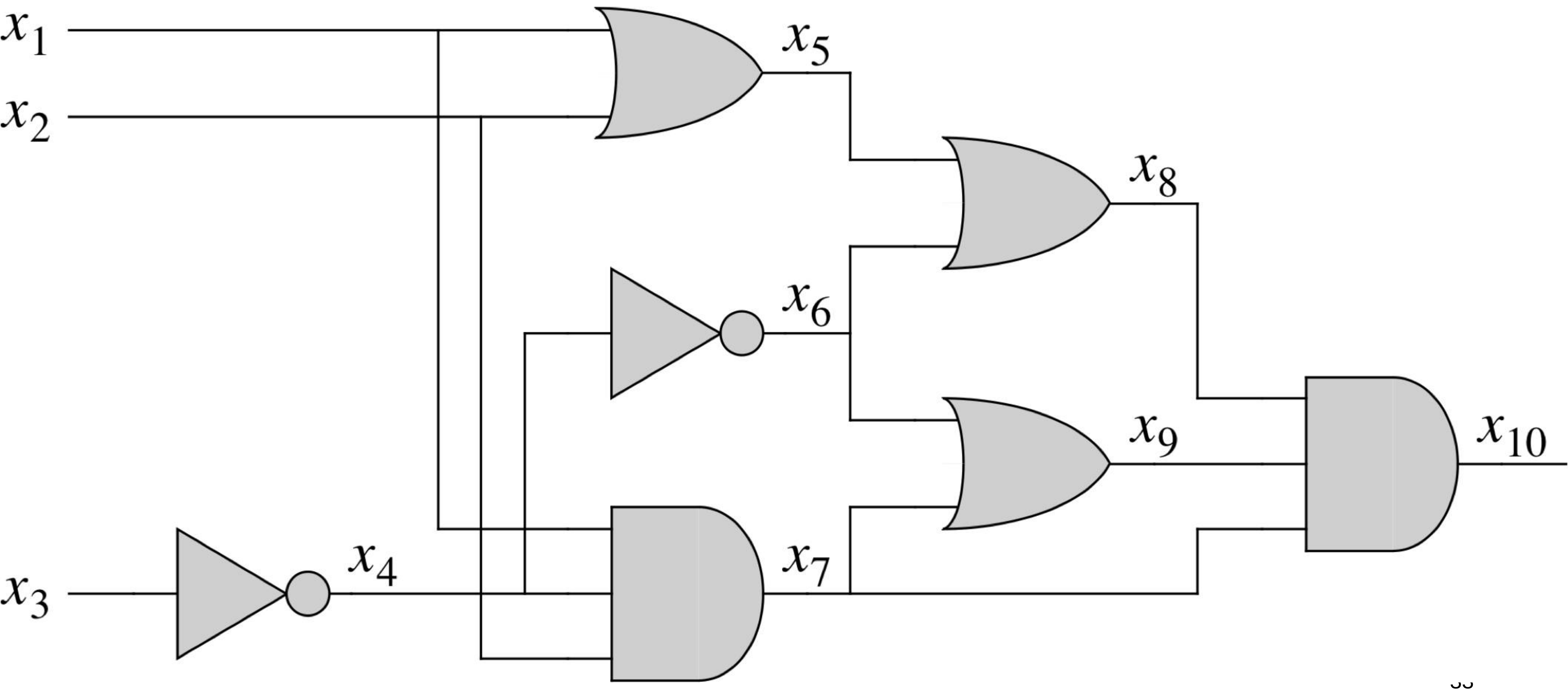
Travelling Salesman Movie

- https://www.youtube.com/watch?v=6ybd5rbQ5rU

NP-Complete Problems: Examples

- Example: **SAT Problem**

Given a Boolean circuit S, is there a satisfying assignment for S? (i.e., variable assignment that outputs 1)



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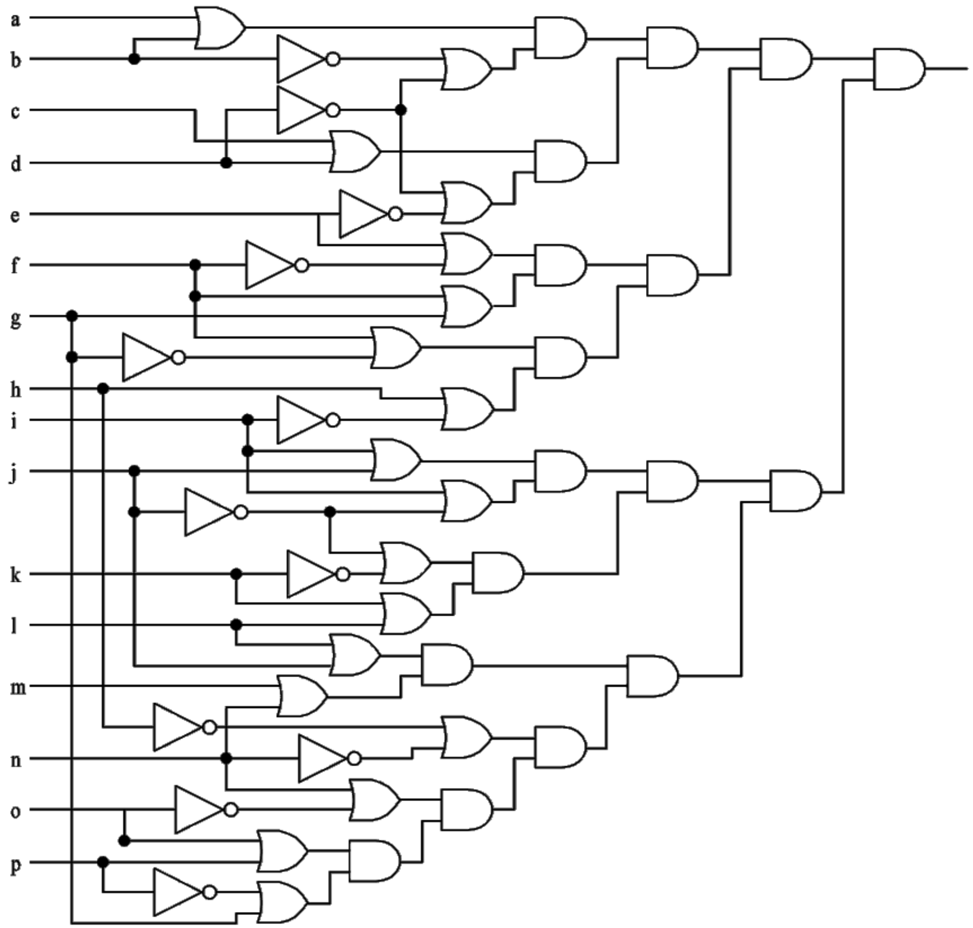

NP-Complete Problems: Examples

- Example: **3-CNF Problem**

Given a Boolean circuit S in 3-CNF form, is there a satisfying assignment for S? (i.e., variable assignment that outputs 1)

- 3-CNF formula: a set ANDed Boolean clauses, each with 3 ORed literals (Boolean variables)

- Example: $\vee$ = OR, $\wedge$ = AND, $\neg$ = NOT
  $(x1 \vee \neg x2 \vee \neg x3) \wedge (\neg x1 \vee x2 \vee x3) \wedge (x1 \vee x2 \vee x3)$

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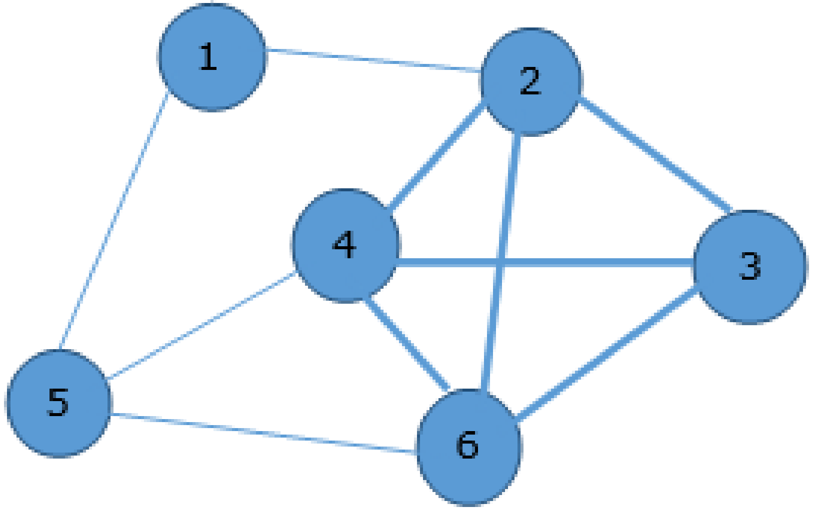
- Solution: $O(k2^n)$ for k clauses and n variables

NP-Complete Problems: Examples

- Example: **(Max) Clique Problem**

Given a graph G=(V,E), find the clique of maximum size.

Clique: fully connected subgraph.

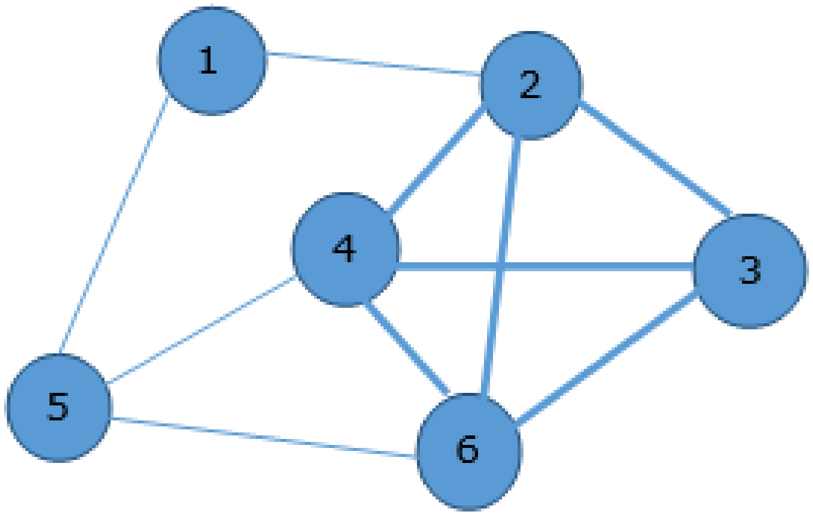


NP-Complete Problems: Examples

- Example: **(Max) Clique Problem**

Given a graph G=(V,E) of n vertices, find the clique of maximum size.

Clique: fully connected subgraph.



- Solution:
  - Assume max clique size k and |V| = n
  - Brute force: $O(n2^n)$
  - Combinations of k: $O(n^k k^2)$
    - Try for k=3,4,5,…
    - k is not constant, so this is not polynomial

Book Readings

- Ch. 34