CS141: Intermediate Data Structures and Algorithms

Greedy Algorithms

Amr Magdy

Activity Selection Problem

- Given a set of activities $S = \{a_1, a_2, \ldots, a_n\}$ where each activity i has a start time s_i and a finish time f_i , where $0 \leq s_i < f_i < \infty$.
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- Two activities are said to be **compatible** if they **do not overlap**.

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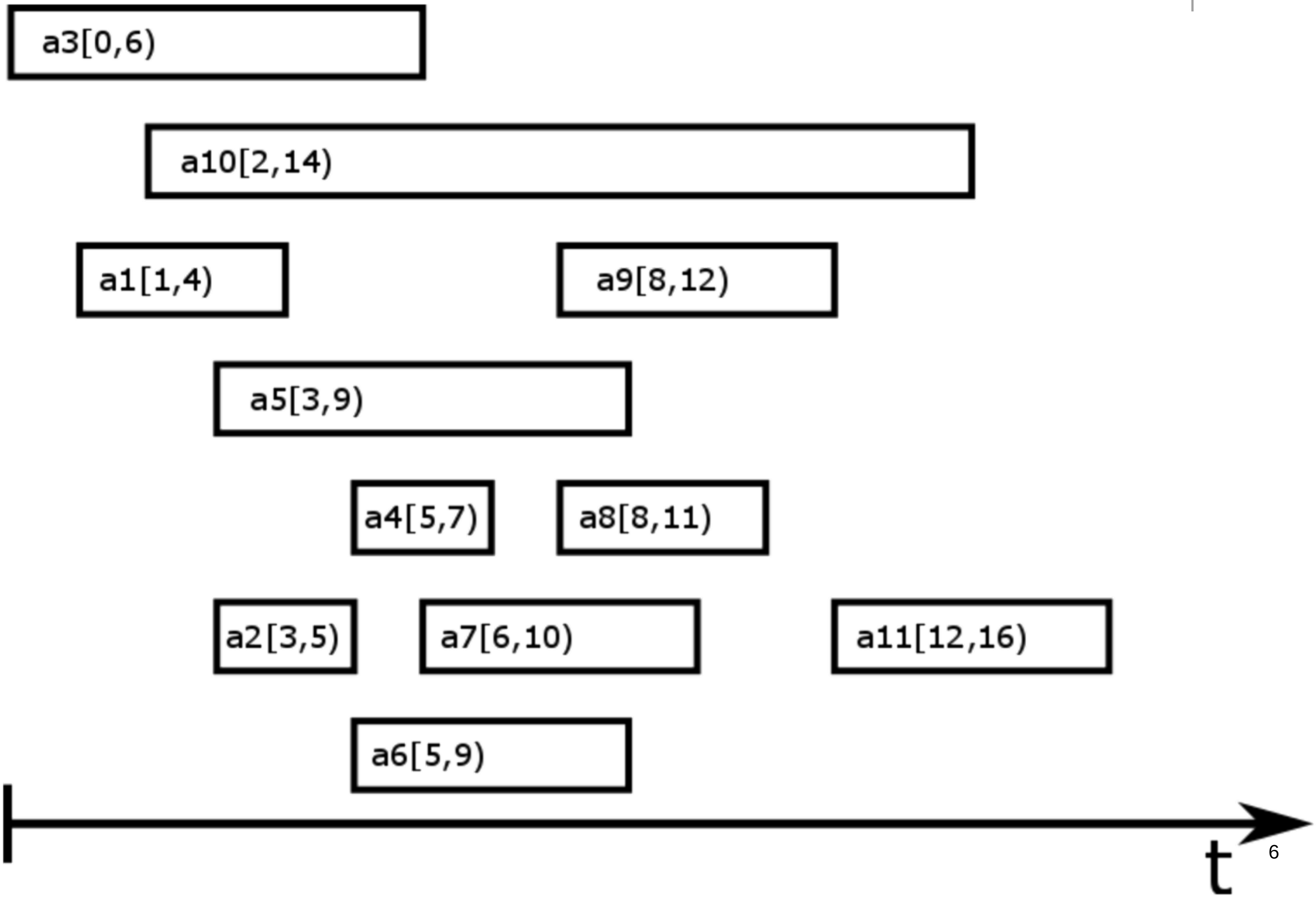
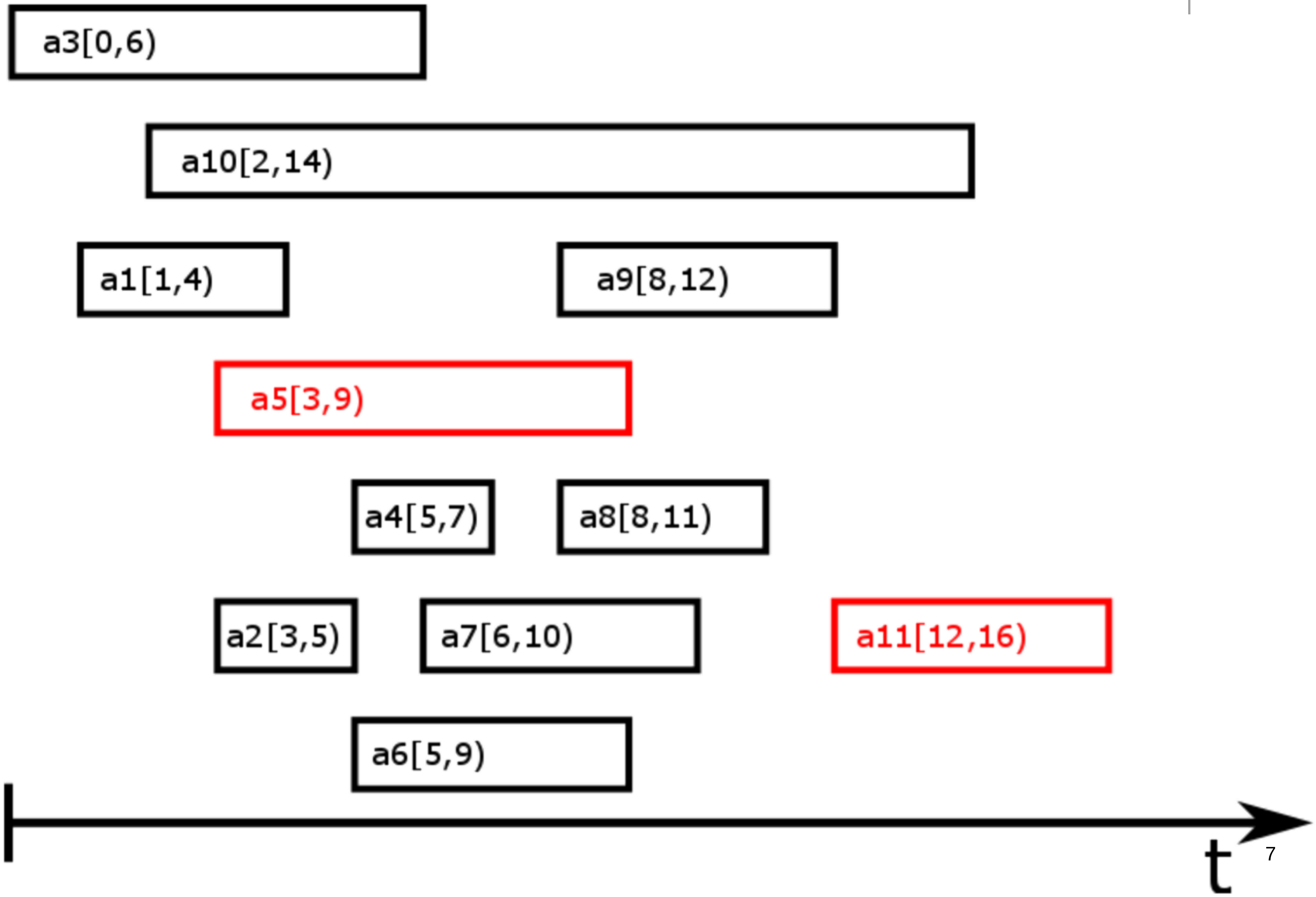
- An activity a_i happens in the half-open time interval $[s_i, f_i)$.
- Activities compete on a single resource, e.g., CPU
- Two activities are said to be **compatible** if they **do not overlap**.
- The problem is to find a **maximum-size compatible subset**, i.e., a one with the maximum number of activities.

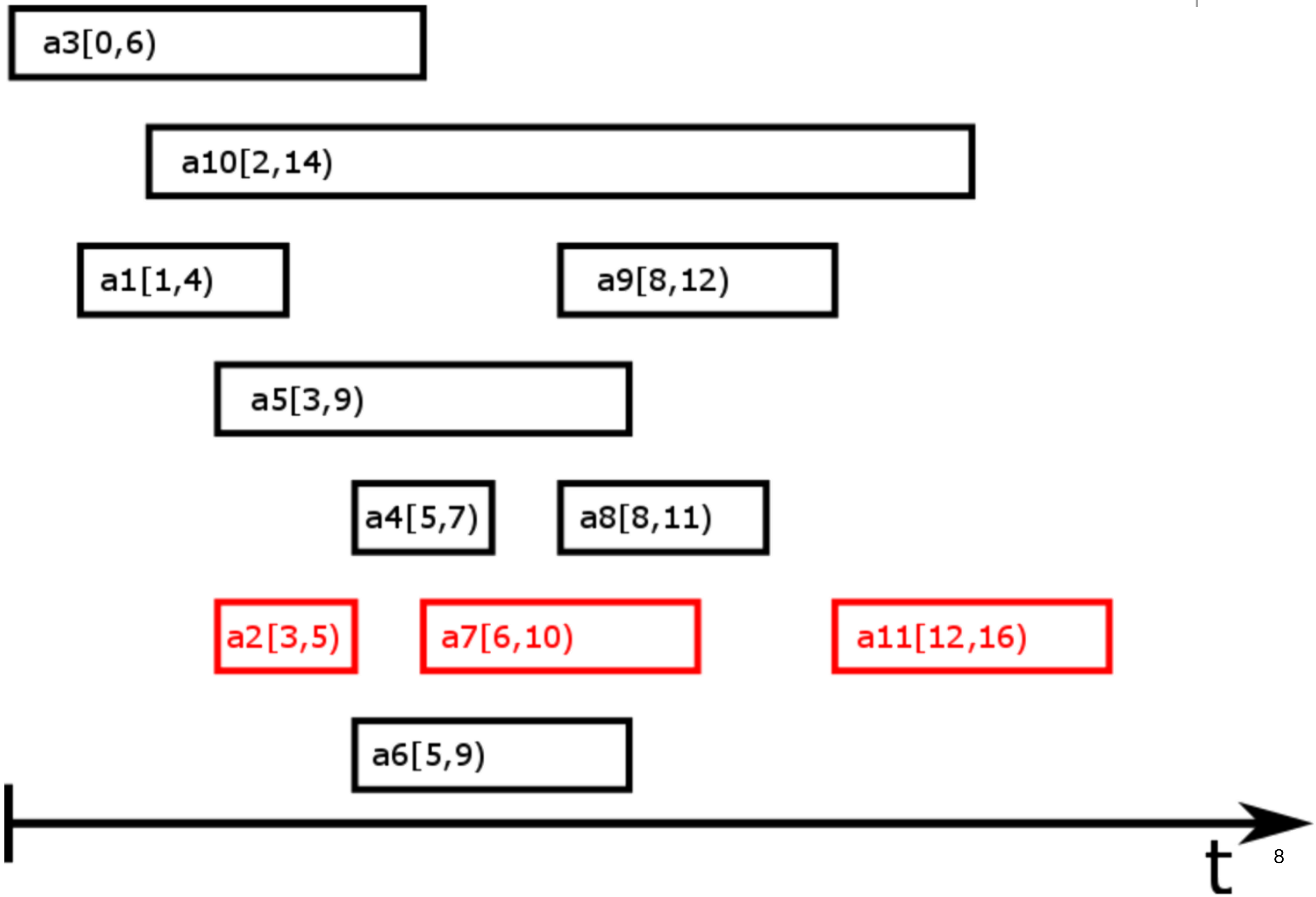
Example


a3[0,6)
a10[2,14)
a1[1,4)
a9[8,12)
a5[3,9)
a4[5,7)
a8[8,11)
a2[3,5)
a7[6,10)
a11[12,16)
a6[5,9)
t

A Compatible Set



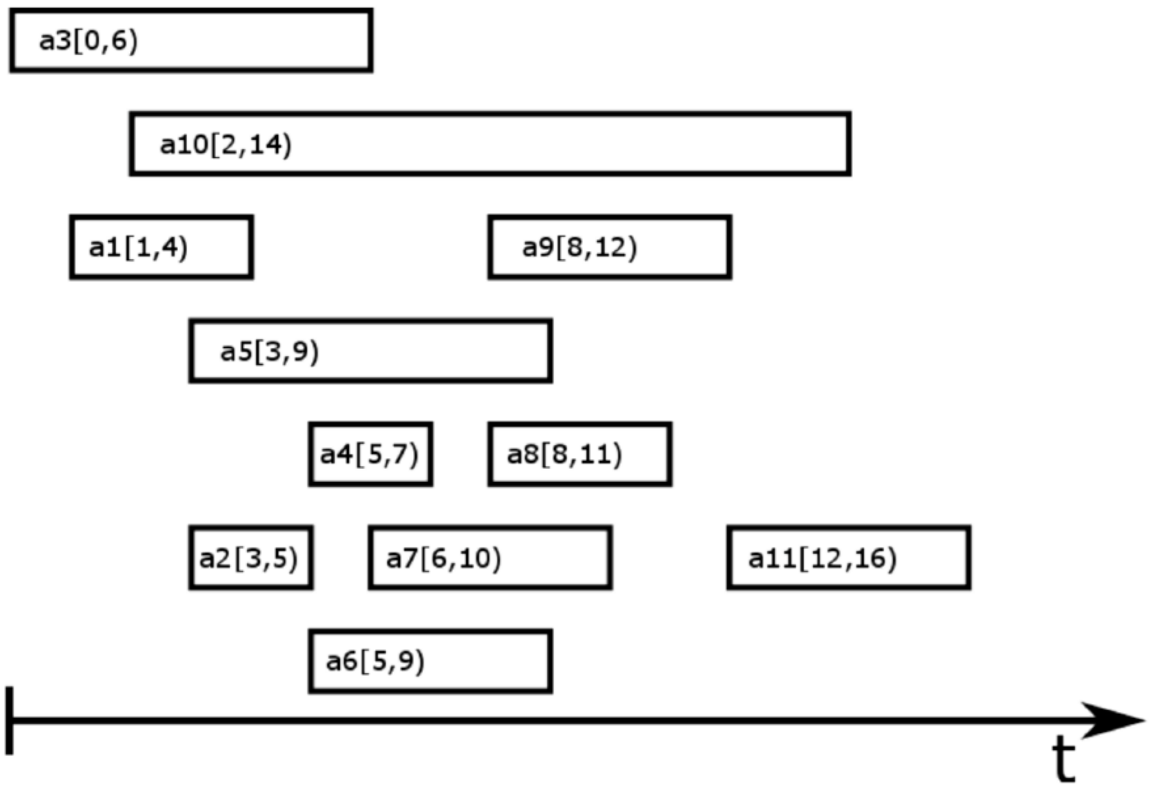
A Better Compatible Set



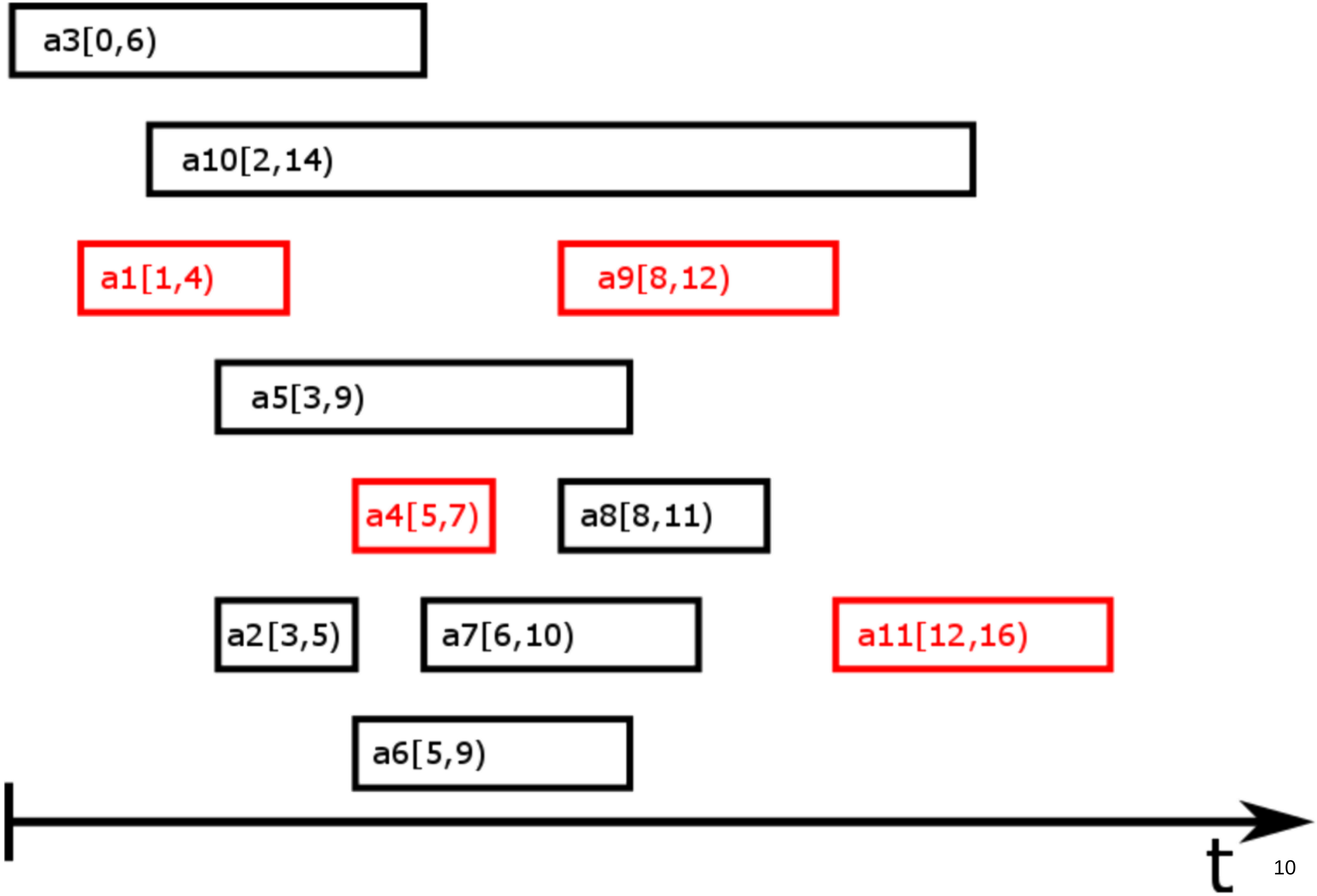
Participation Exercise

Name: Student ID:

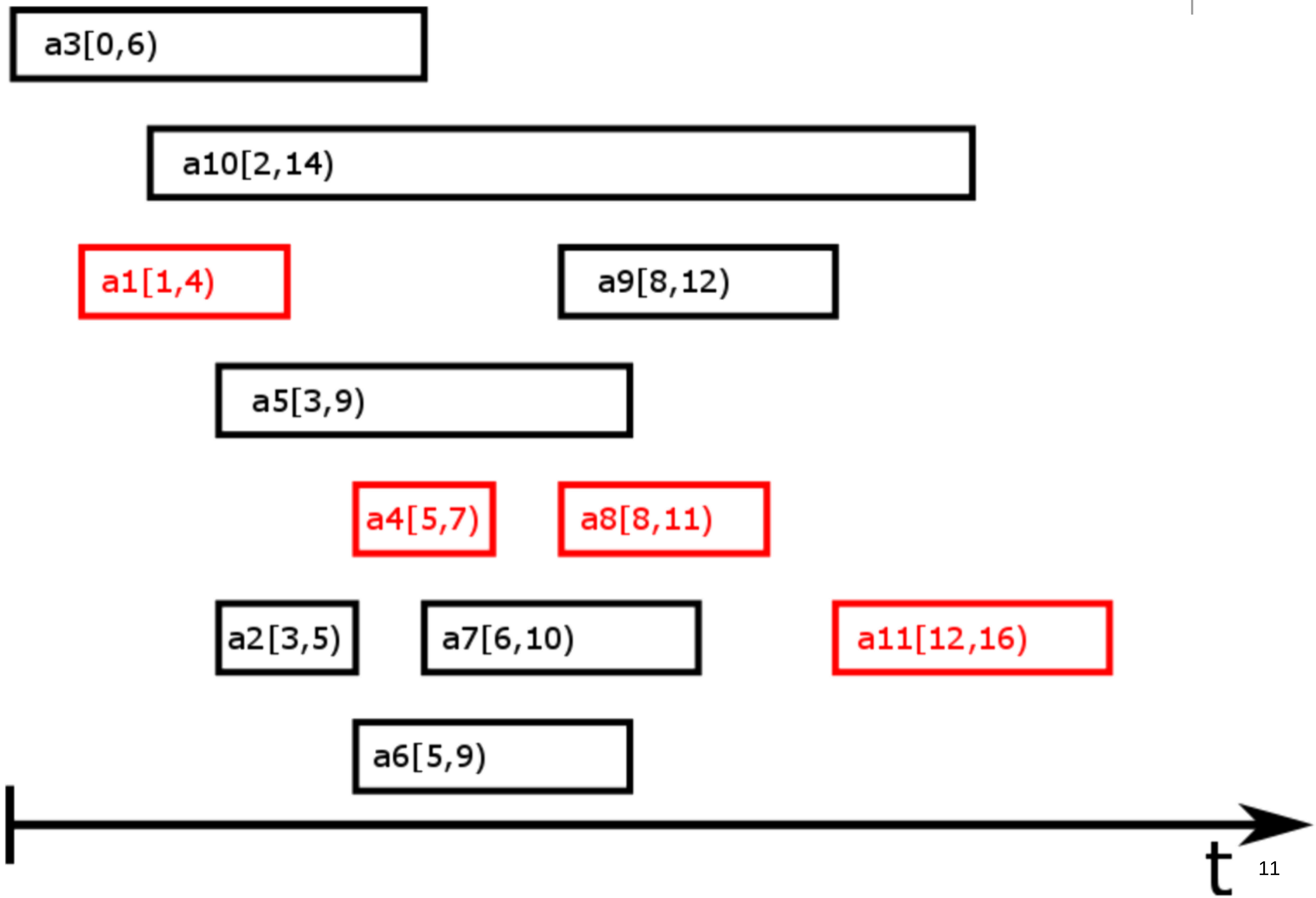
Two activities are said to be **compatible** if they **do not overlap**. For example, a1 and a8 are compatible activities. The subset that contains a2, a6, and a11 is a compatible subset of size 3, i.e., has three compatible activities. Find a **maximum-size compatible subset**, i.e., a one with the maximum number of activities, of the following set of activities.



An Optimal Solution



Another Optimal Solution



Activity Selection Problem

- Solution algorithm?
  - Brute force (naïve): all possible combinations → $O(2^n)$
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  - i.e., take the best choice so far, reduce the problem size, and solve a subproblem later

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- Instead, can we make a **greedy** choice?
  - i.e., take the best choice so far, reduce the problem size, and solve a subproblem later
- Greedy choices
  - Longest first
  - Shortest first
  - Earliest start first
  - Earliest finish first
  - …?

Activity Selection Problem

- Greedy choice: shortest length first
  - Why? To accommodate as much activities as possible

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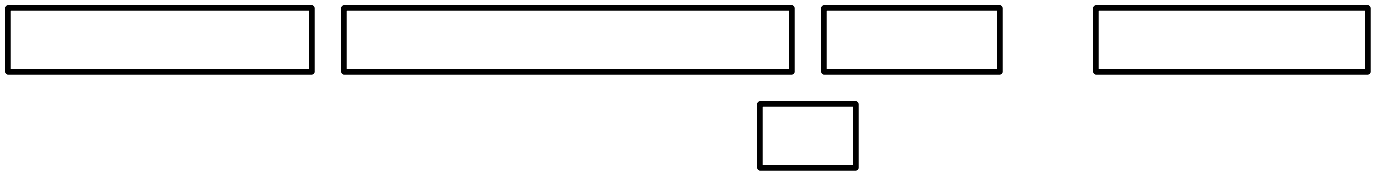

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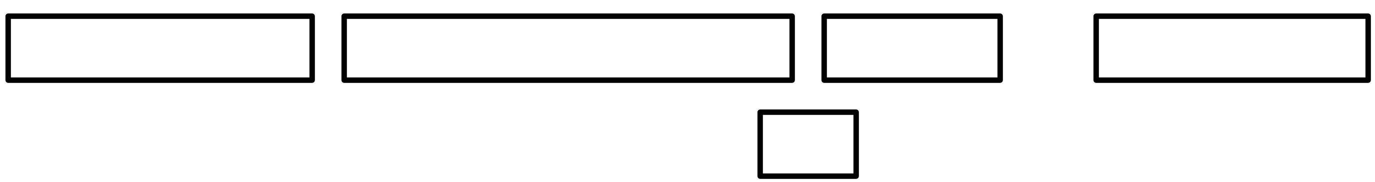

- This greedy choice does not work

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  - Why? It leaves as much resource as possible for other tasks
- Is this choice correct? Does it guarantee an optimal solution?
  - Can we find a counter example against this choice?
    - Not clear
    - Let’s try to prove its correctness, if we cannot, then it is wrong

Activity Selection Problem

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- Greedy choice property
  - Prove that if a_m has the earliest finish time, it must be included in some optimal solution.

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- Assume a set S and a solution set A, where $a_m \notin A$
  - Let a_j is the activity with the earliest finish time in A (not in S)
  - Compose another set $A' = A - \{a_j\} \cup \{a_m\}$
  - A' still have all activities disjoint (as a_m has the global earliest finish time and A activities are already disjoint), and $|A'|=|A|$
  - Then A' is an optimal solution
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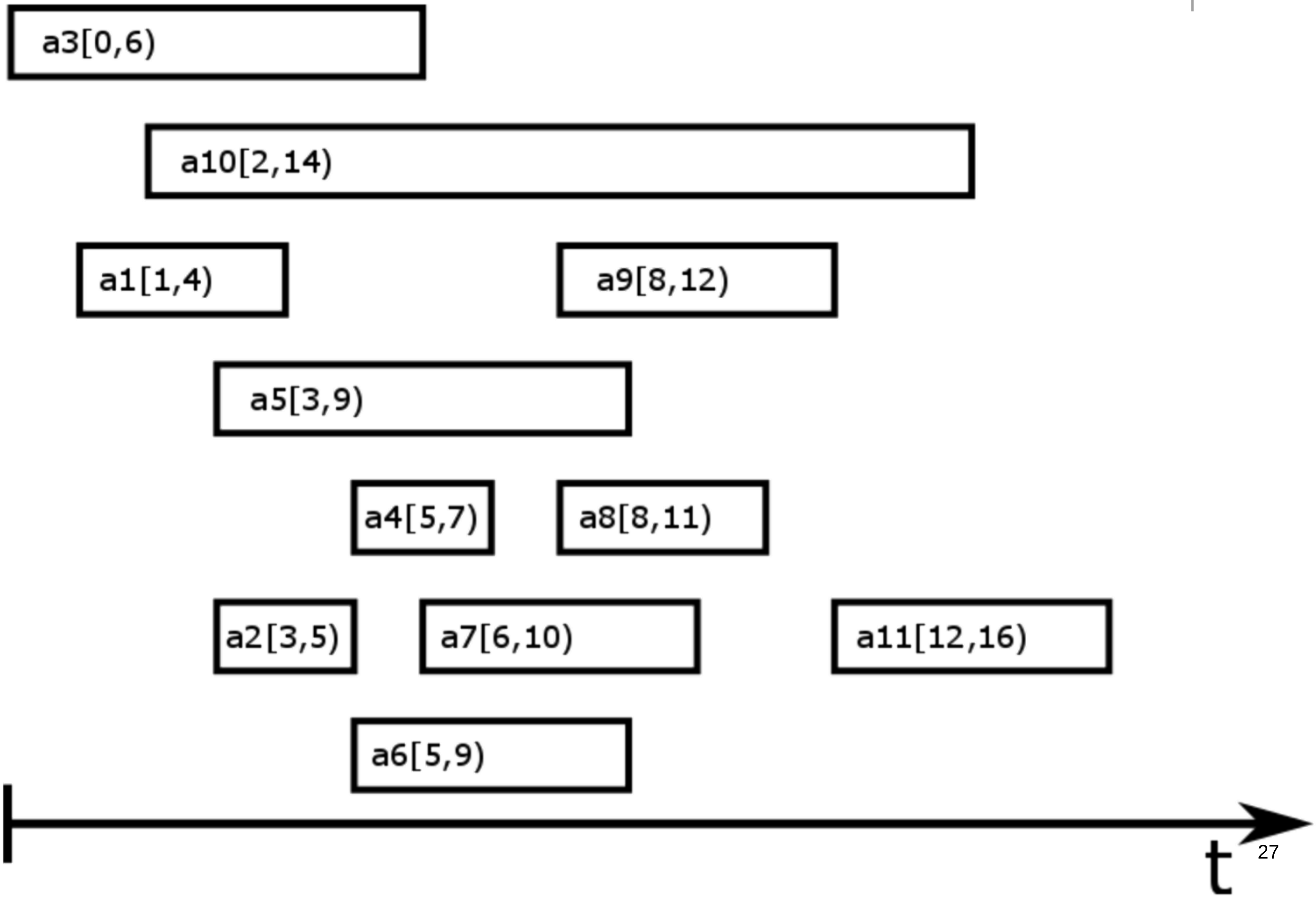
- In the example:
  - A ={a2, a4, a8, a11}, am= a1, aj= a2
  - A'={a1, a4, a8, a11}
  - As a1 finishes before a2, then a1 is compatible with a4, a8, a11

Activity Selection Problem

- Solution:
  - Include earliest finish activity a_m in solution A
  - Remove all a_m's incompatible activities
  - Repeat for the remaining earliest finish activity

Activity Selection Problem: Greedy Solution



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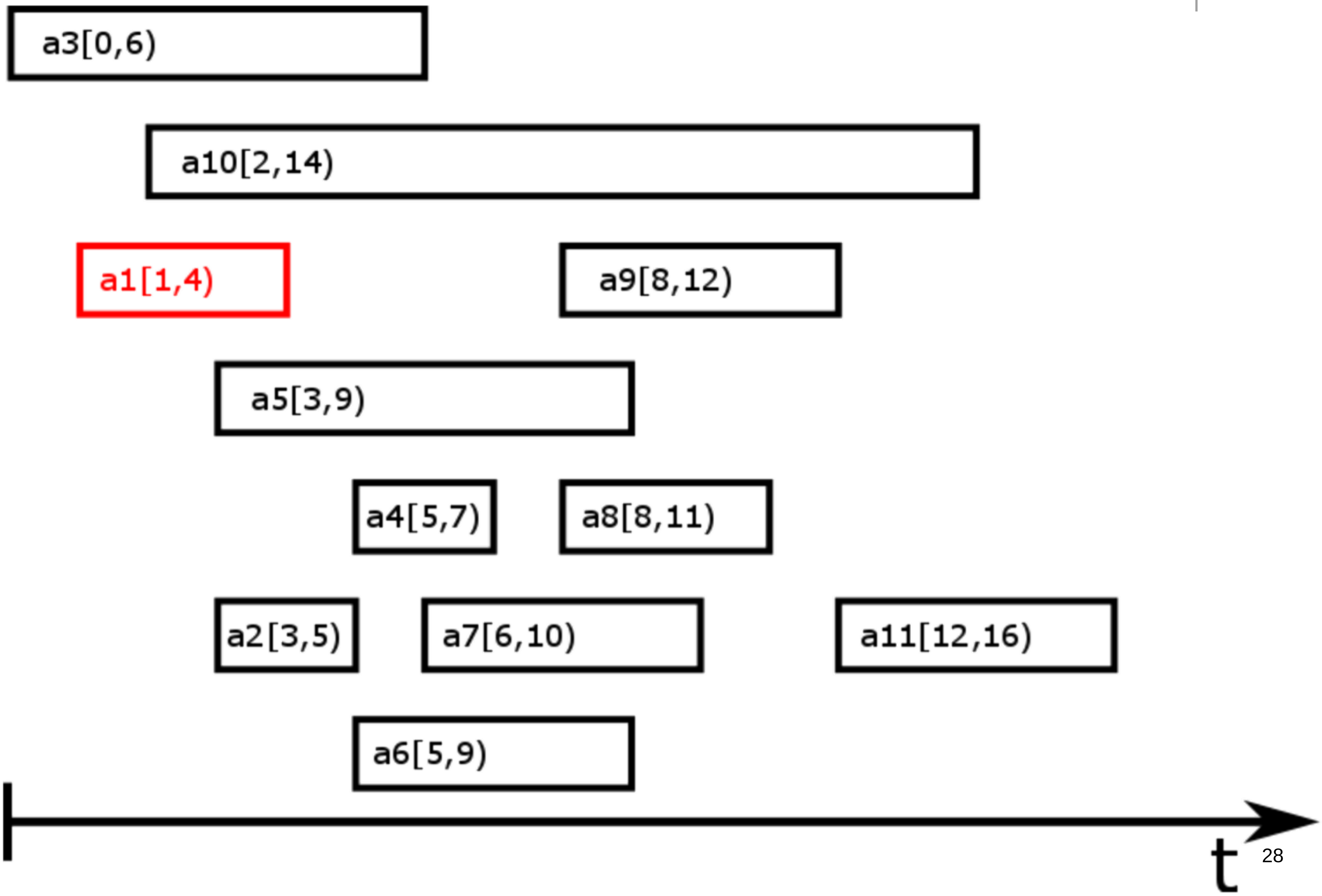

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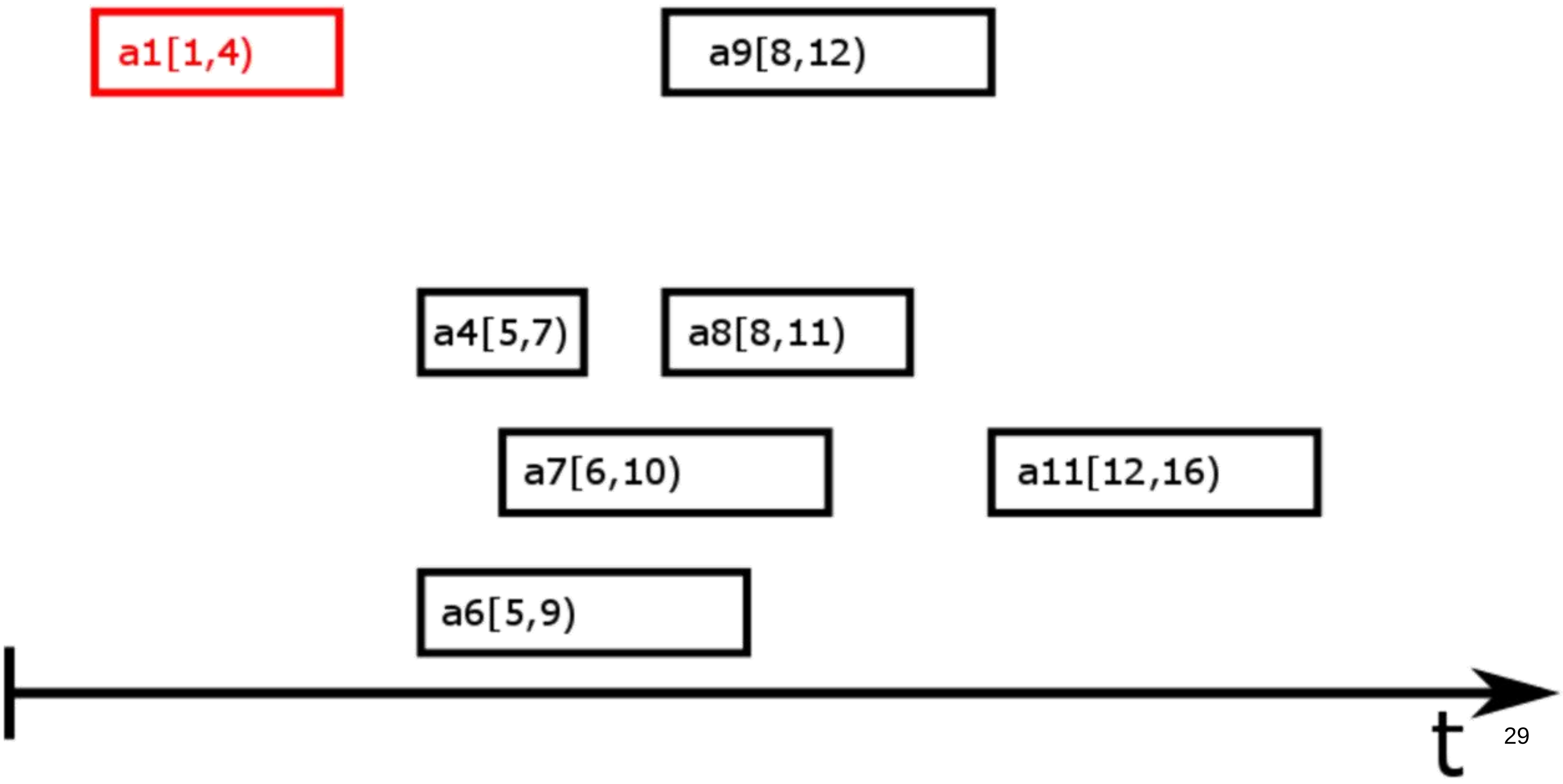

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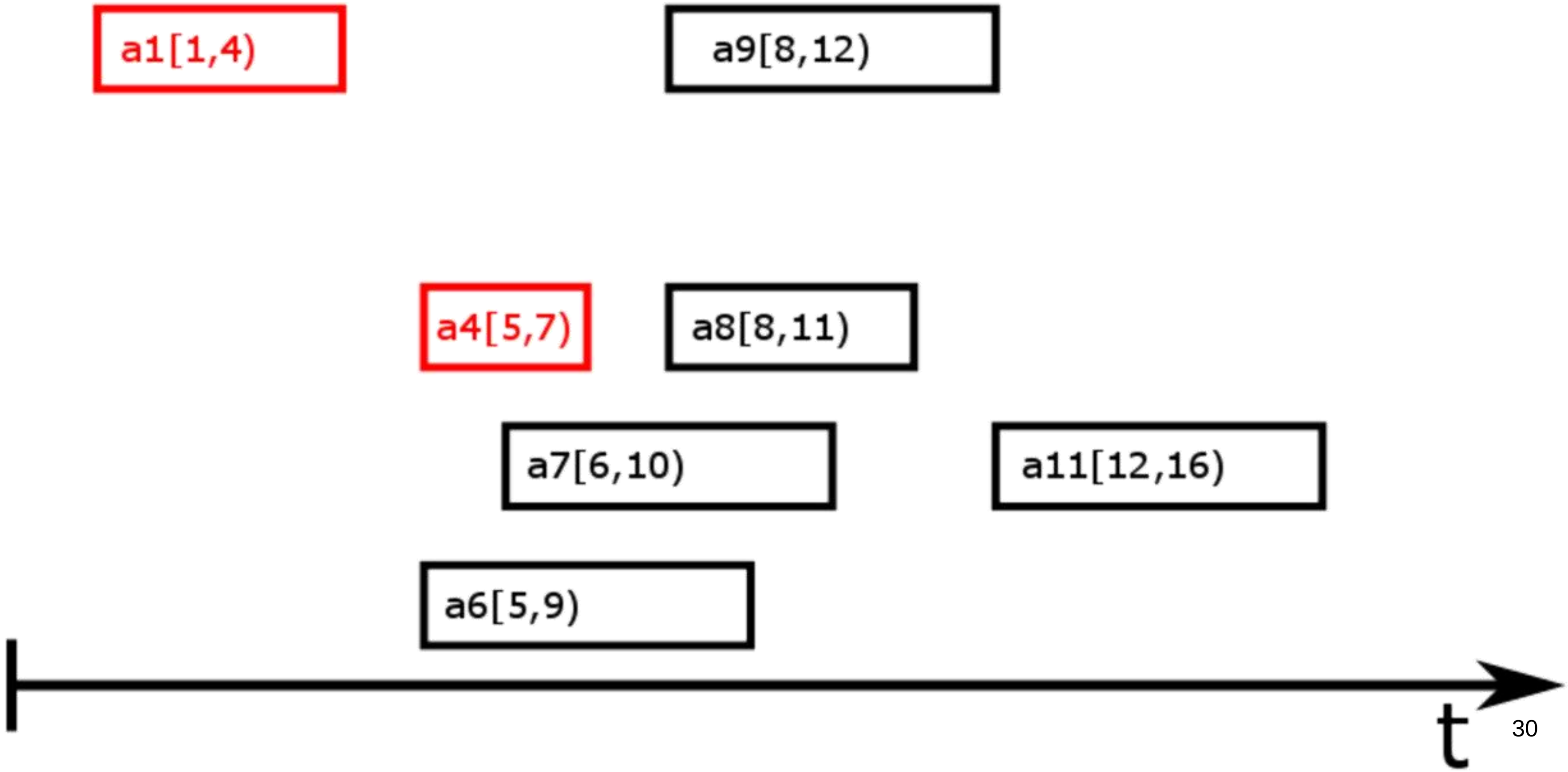

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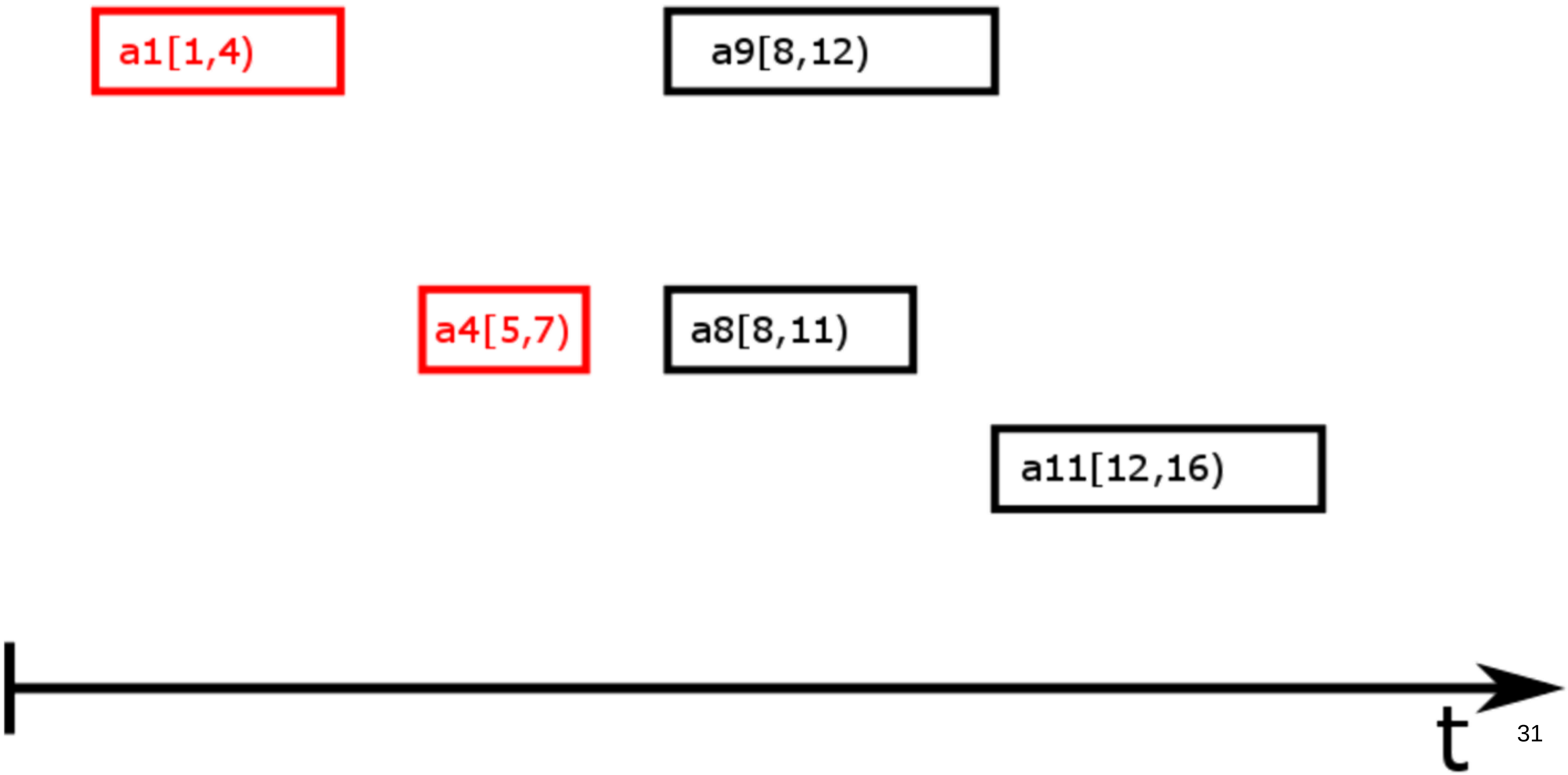

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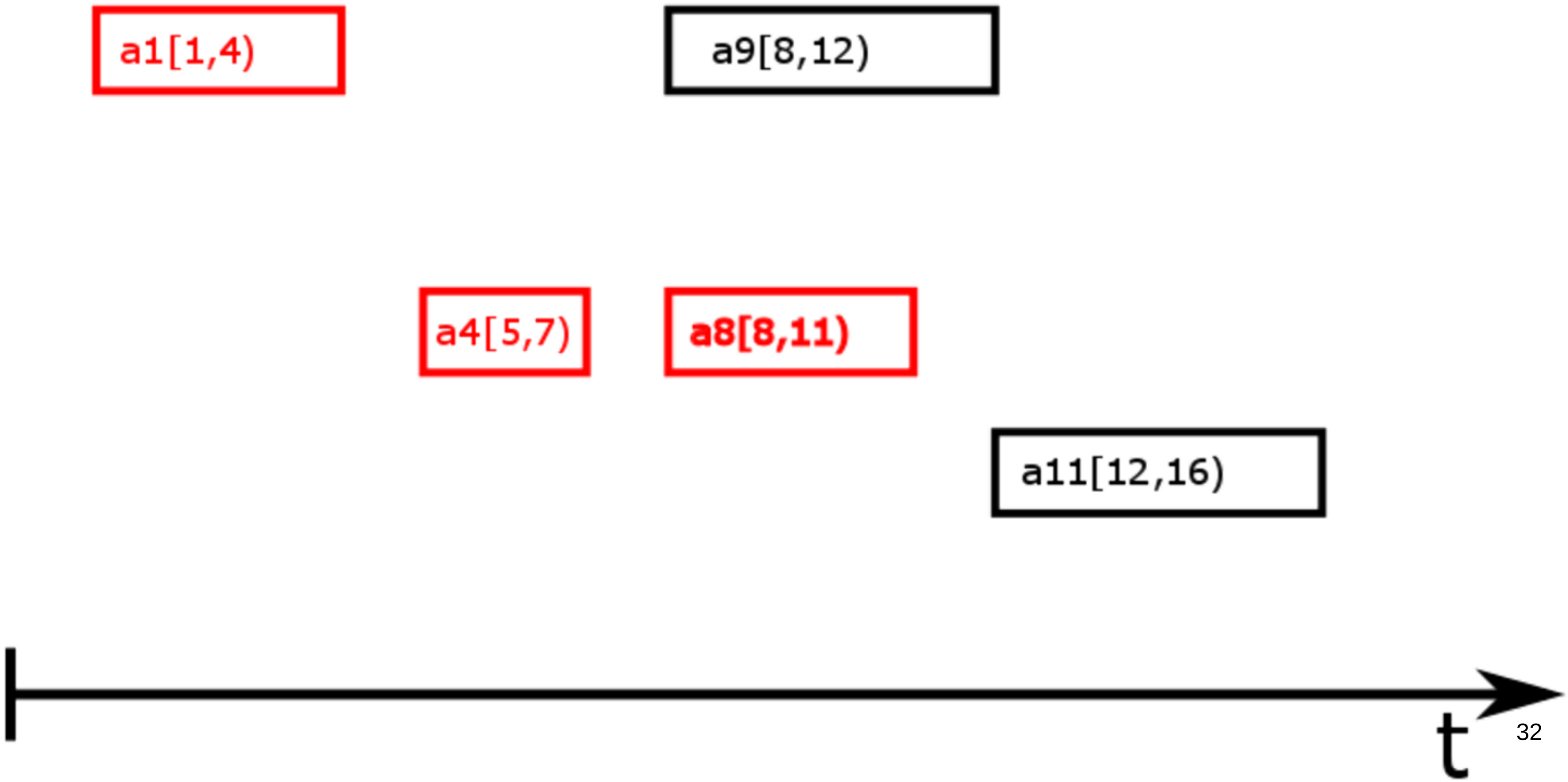

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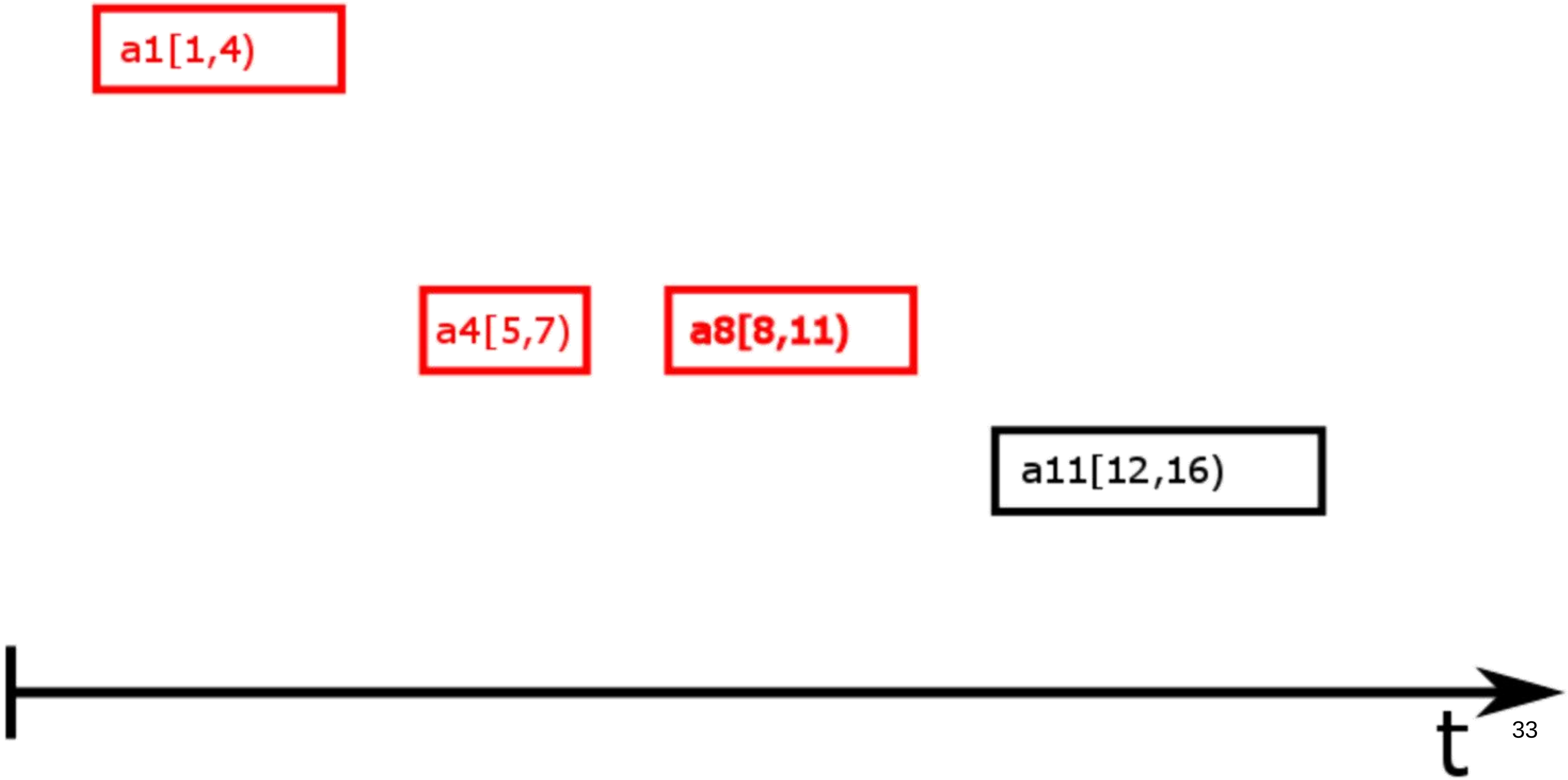

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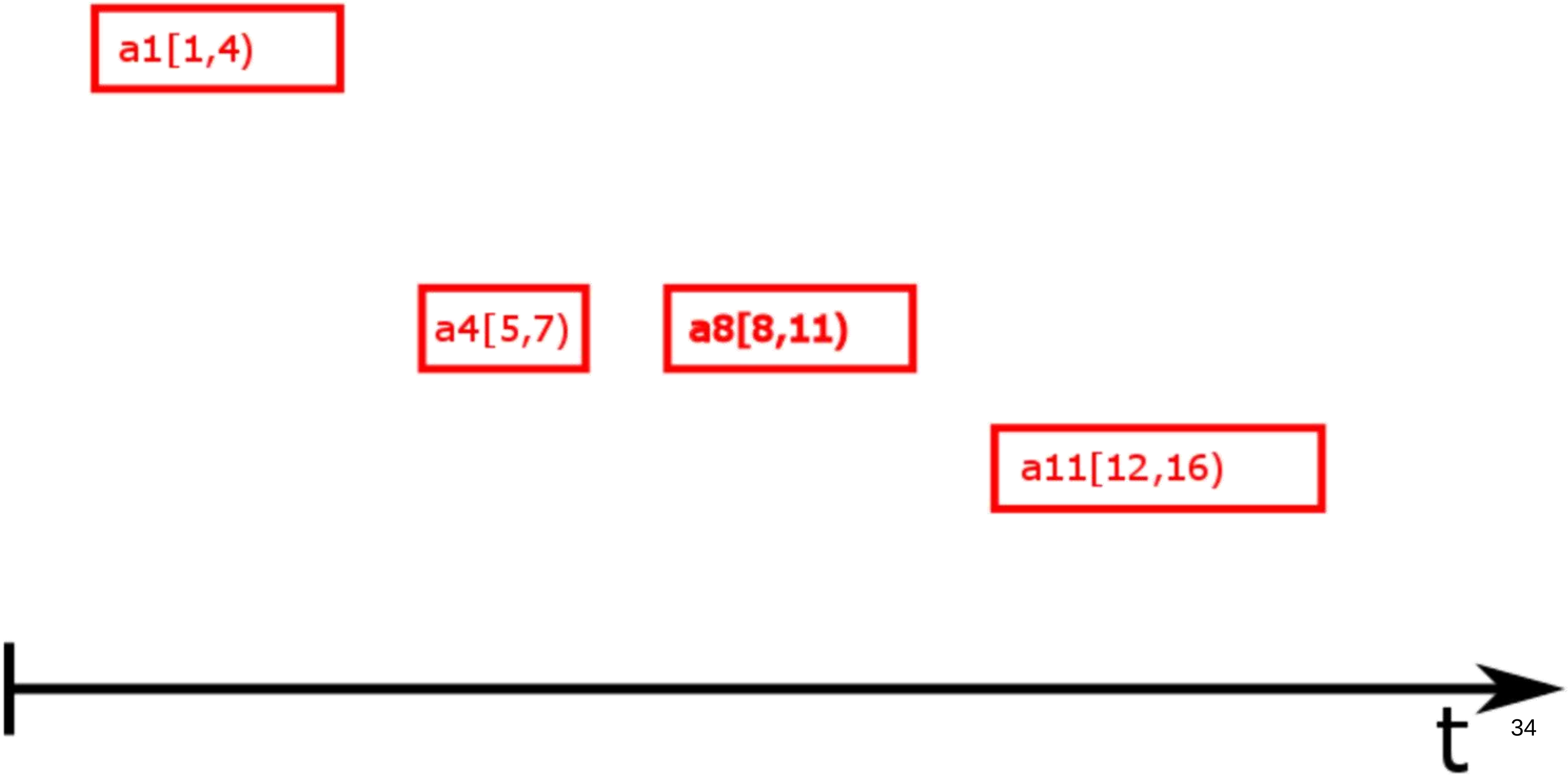

Activity Selection Problem

- Pseudo code?

Activity Selection Problem

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```
findMaxSet(Array a, int n)
{
        - Sort "a" based on earliest finish time
        - result ← {}
        - for i = 1 to n
                validAi = true
                 for j = 1 to result.size
                        if (a[i] is incompatible with result[j])
                                validAi = false
                if (validAi)
                        result ← result U a[i]
        - return result
}
```

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  - i.e., the optimal solution of a bigger problem has optimal solutions for subproblems

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- Assume A is an optimal solution for S
  - Is A' = A-{a_i} an optimal solution for S' = S-{a_i and its incompatible activities}?
  - If A' is not an optimal solution, then there an optimal solution A'' for S' so that |A''| > |A'|
  - Then B=A'' U {a_i} is a solution for S, |B|=|A''|+1, |A|=|A'|+1
  - Then |B| > |A|, i.e., |A| is not an optimal solution, **contradiction**
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- Why optimal substructure?
  - To guarantee it is correct to use solutions of subproblems after applying greedy choice

Elements of a Greedy Algorithm

1. Optimal Substructure
2. Greedy Choice Property

Greedy vs. Dynamic Programming

- Solving the bigger problem include
  One choice (greedy) vs Multiple possible choices

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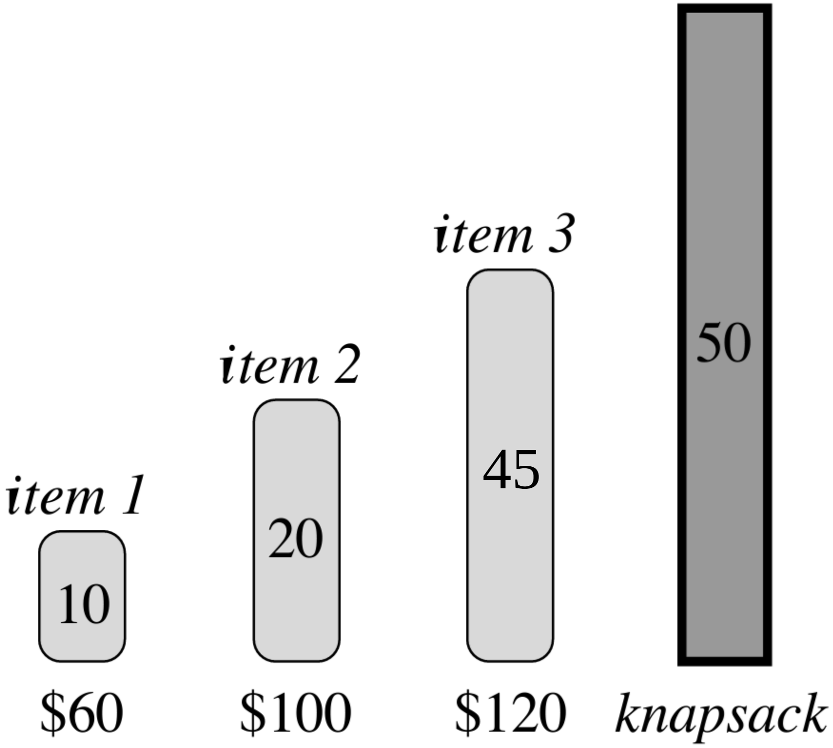
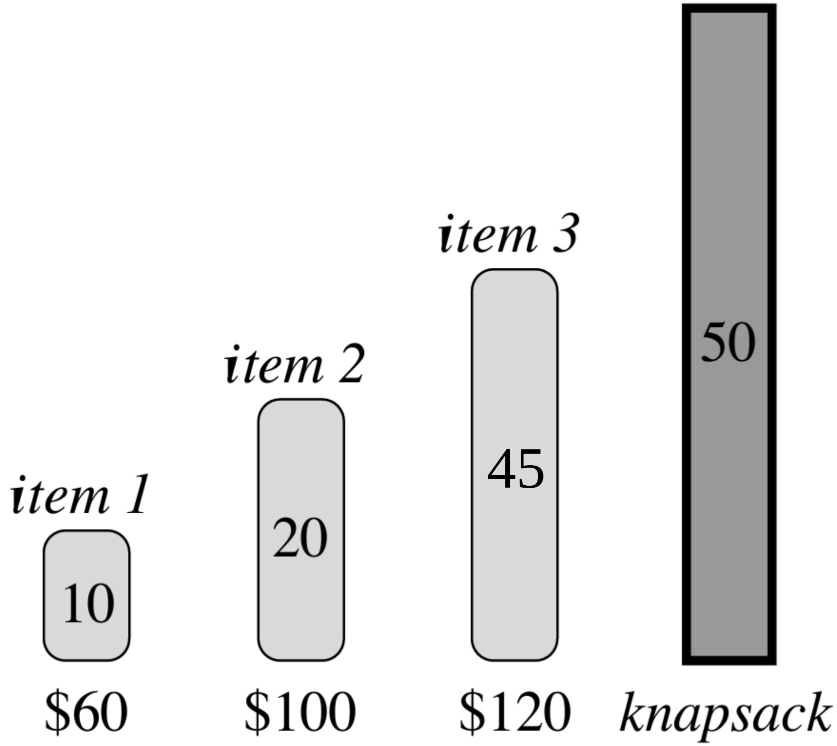
- Elements:

Greedy	DM
Optimal substructure	Optimal substructure
Greedy choice property	Overlapping subproblems

Knapsack Problem

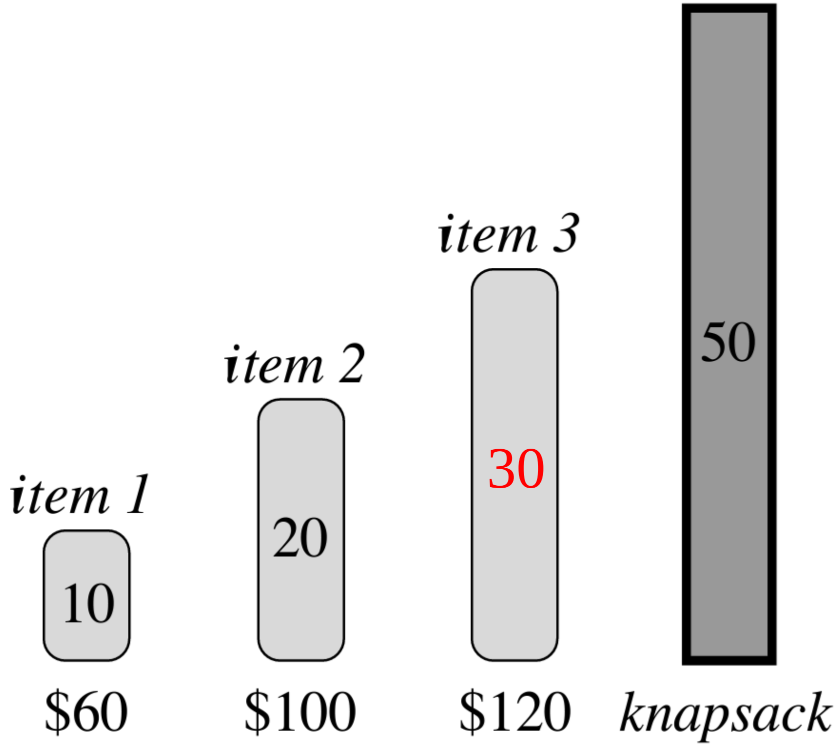


Knapsack Problem



- 0-1 Knapsack: Each item either included or not
- Greedy choices:
  - Take the most valuable → Does not lead to optimal solution
  - Take the most valuable per unit → Works in this example

Knapsack Problem



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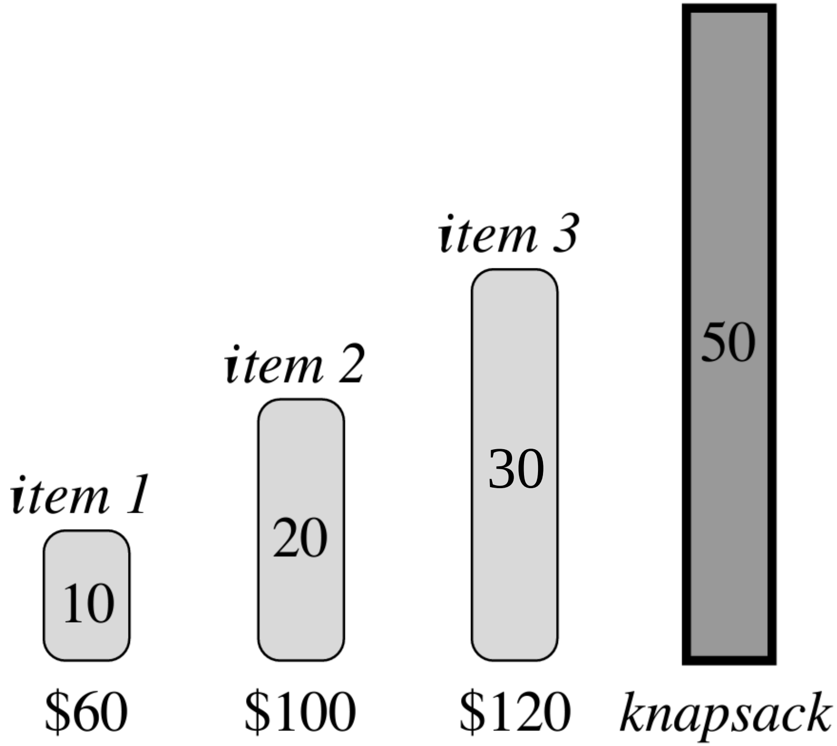
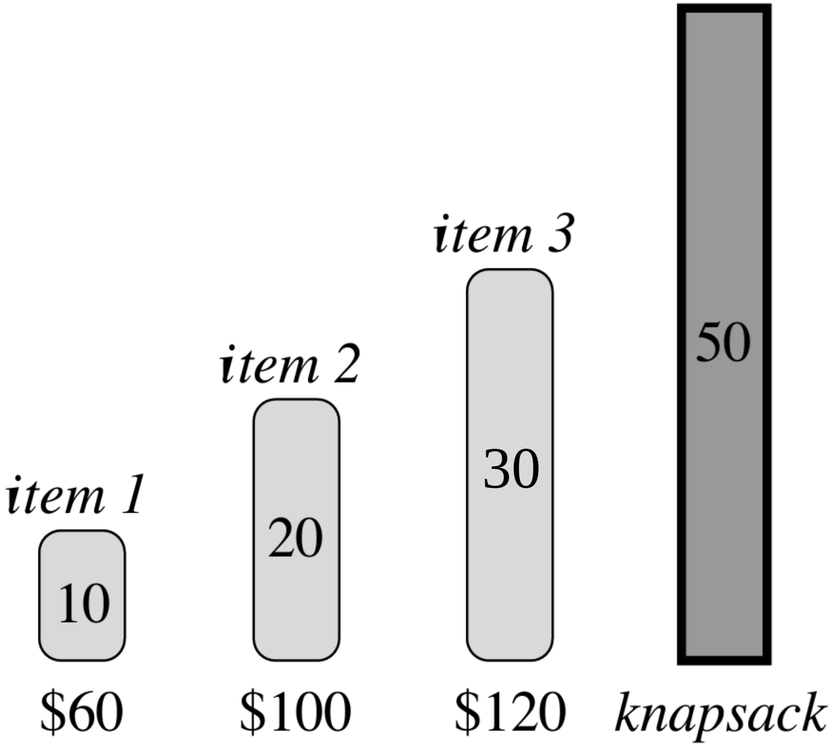


- Fractional Knapsack: Part of items can be included

Knapsack Problem



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Fractional Knapsack Problem

- Greedy choice property: take the most valuable per weight unit

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- Greedy choice property: take the most valuable per weight unit
- Proof of optimality:
  - Given the set S ordered by the value-per-weight, taking as much as possible x_j from the item j with the highest value-per-weight will lead to an optimal solution X
  - Assume we have another optimal solution $X`$ where we take less amount of item $j$, say $x_j` < x_j$.
  - Since $x_j` < x_j$, there must be another item $k$ which was taken with a higher amount in $X`$, i.e., $x_k` > x_k$.
  - We create another solution $X``$ by doing the following changes in $X`$
    - Reduce the amount of item k by a value z and increase the amount of item j by a value z
    - The value of the new solution $V`` = V` + z\ v_j/w_j - z\ v_k/w_k$ $= V` + z\ (v_j/w_j - v_k/w_k) \rightarrow v_j/w_j - v_k/w_k \geq 0 \rightarrow V`` \geq V`$

Fractional Knapsack Problem

- Optimal substructure

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- Optimal substructure
- Given the problem S with an optimal solution X with value V, we want to prove that the solution $X` = X - x_j$ is optimal to the problem $S` = S$ - {j} and the knapsack capacity $W` = W - x_j$
- Proof by contradiction
  - Assume that $X`$ is not optimal to $S`$
  - There is another solution $X``$ to $S`$ that has a higher total value $V`` > V`$
  - Then $X``$ U $\{x_j\}$ is a solution to $S$ with value $V``+ x_j > V`+ x_j > V$
  - Contradiction as V is the optimal value

Fractional Knapsack Problem

```
Fknapsack (W, S, v's, w's) {
        - Sort S based on vi/wi value
        - rw = W
        - result = { }
        - for each si in S
                if(wi <= rw)
                        result = result U si
                        rw = rw-wi
                else
                        result = result U rw/wi * si
                        rw = 0
        - return result
}
```

Huffman Codes

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

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- Example: Message ‘JAVA’ a = “0”, j = “11”, v = “10”
  Encoded message “110100” Decoding “110100”

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- Example: Message ‘JAVA’ a = “0”, j = “11”, v = “10”
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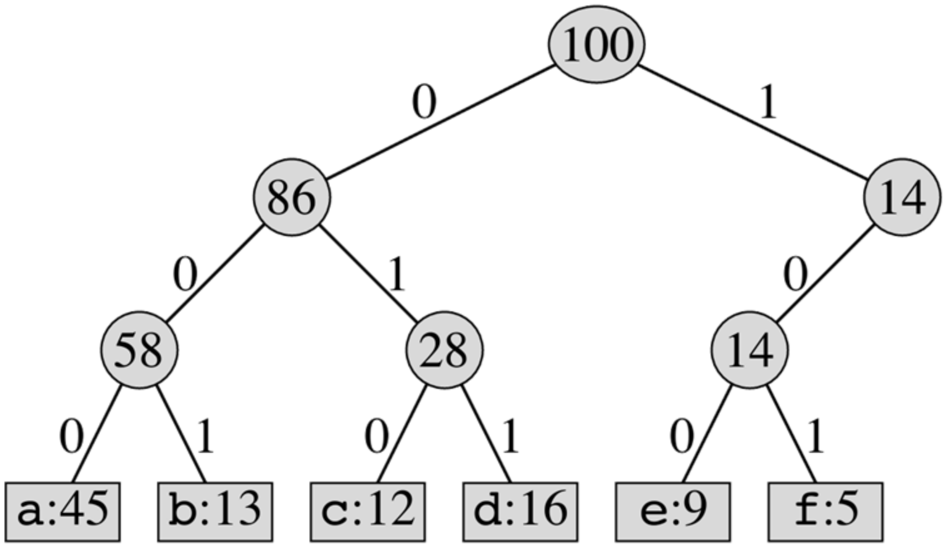
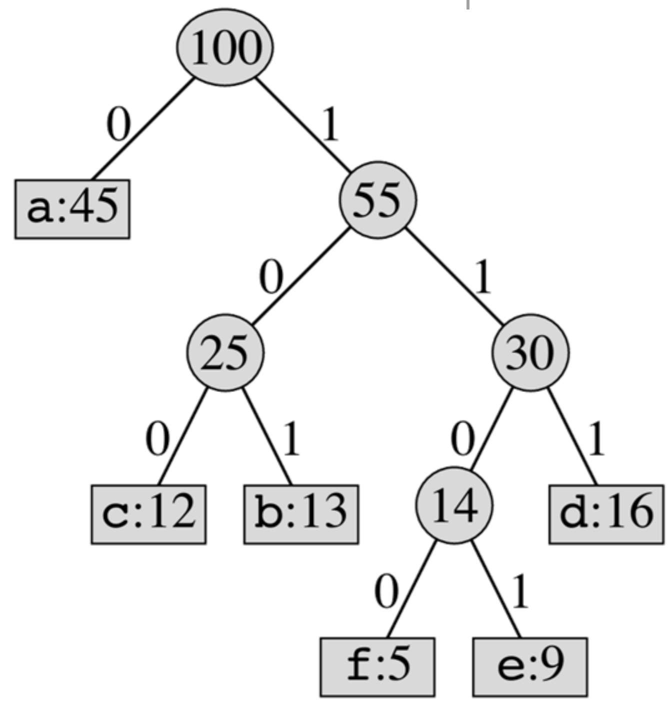
- In the table:
  Encoding with fixed-length needs 300K bits
  Encoding with variable-length needs 224K bits

Huffman Codes

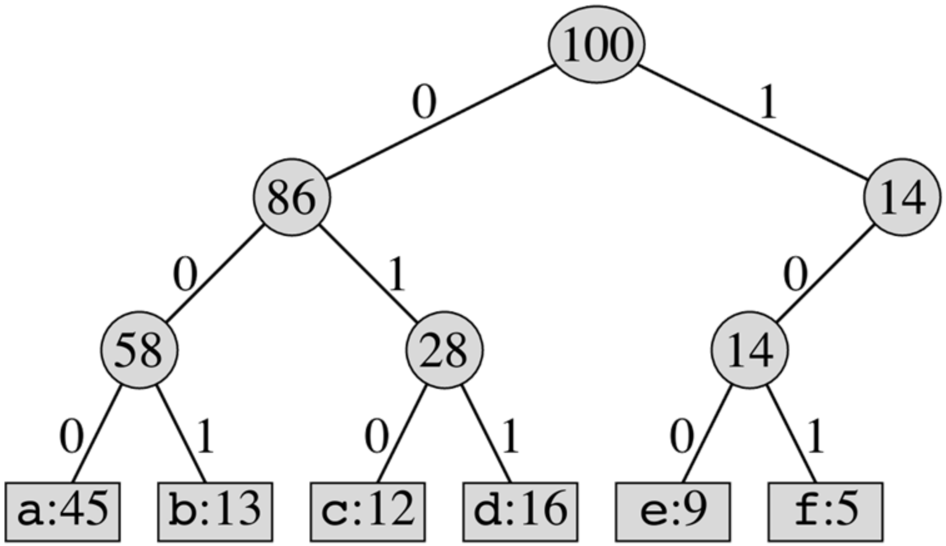


Fixed-length tree

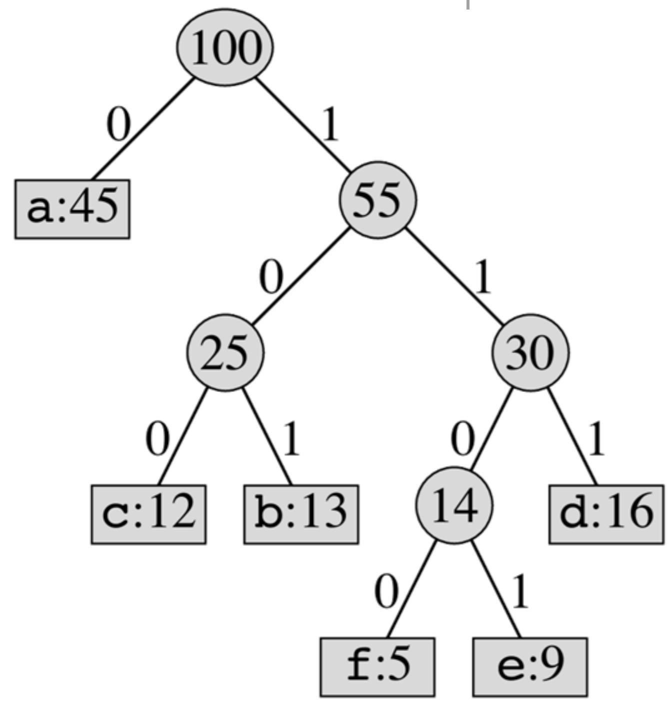


Variable-length tree

Huffman Codes



Fixed-length tree



Variable-length tree

We need an algorithm to build the optimal variable-length tree

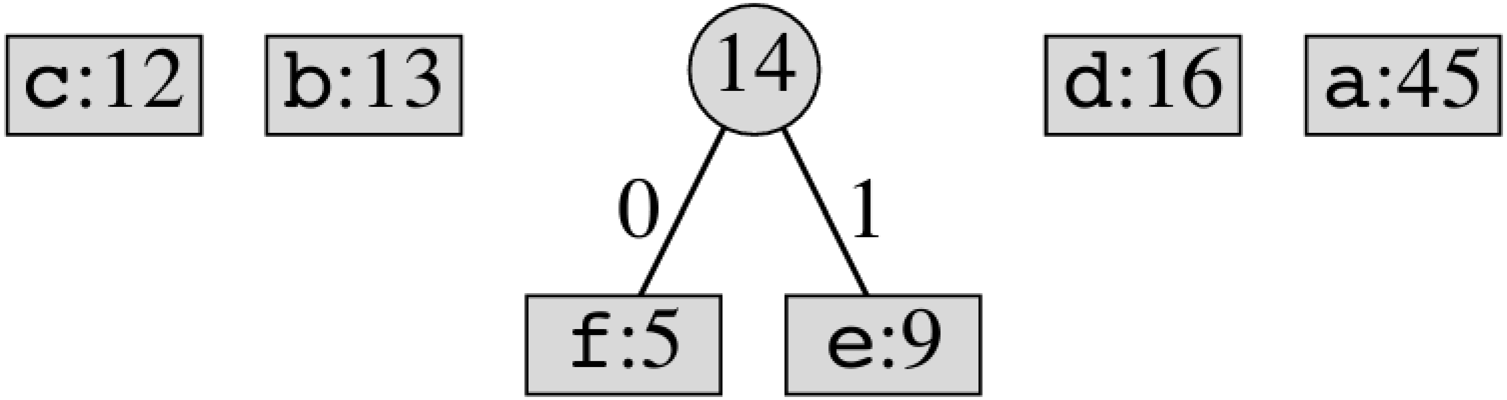
Huffman Codes: Tree Construction

```
HUFFMAN(C)
n = |C|
Q = C
for i = 1 to n − 1
    allocate a new node z
    z.left = x = EXTRACT-MIN(Q)
    z.right = y = EXTRACT-MIN(Q)
    z.freq = x.freq + y.freq
    INSERT(Q, z)
return EXTRACT-MIN(Q)        // return the root of the tree
```

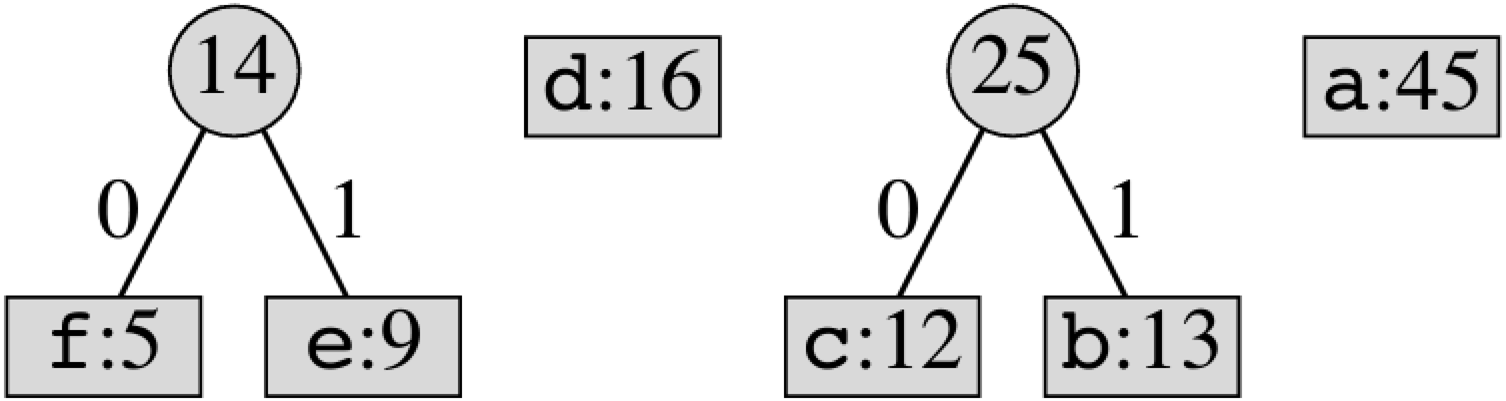
Huffman Codes: Tree Construction

f:5	e:9	c:12	b:13	d:16	a:45

Huffman Codes: Tree Construction



Huffman Codes: Tree Construction



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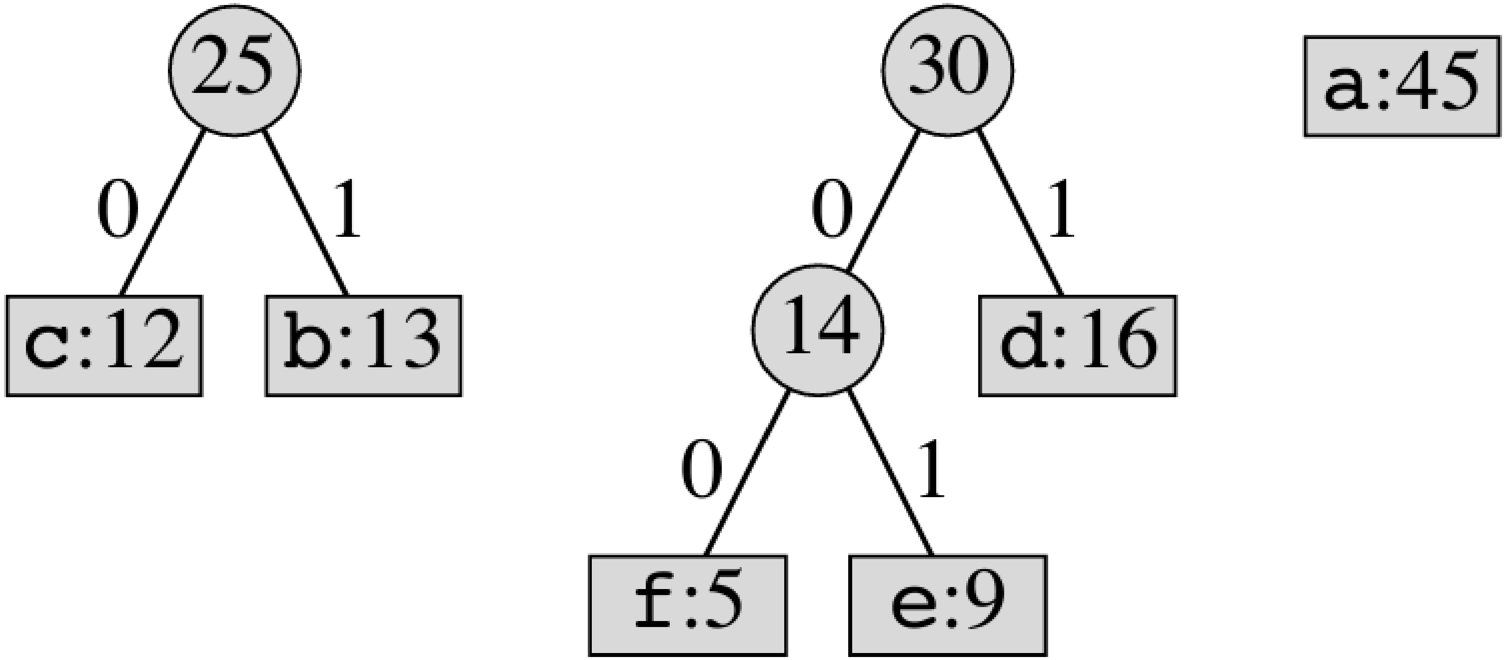

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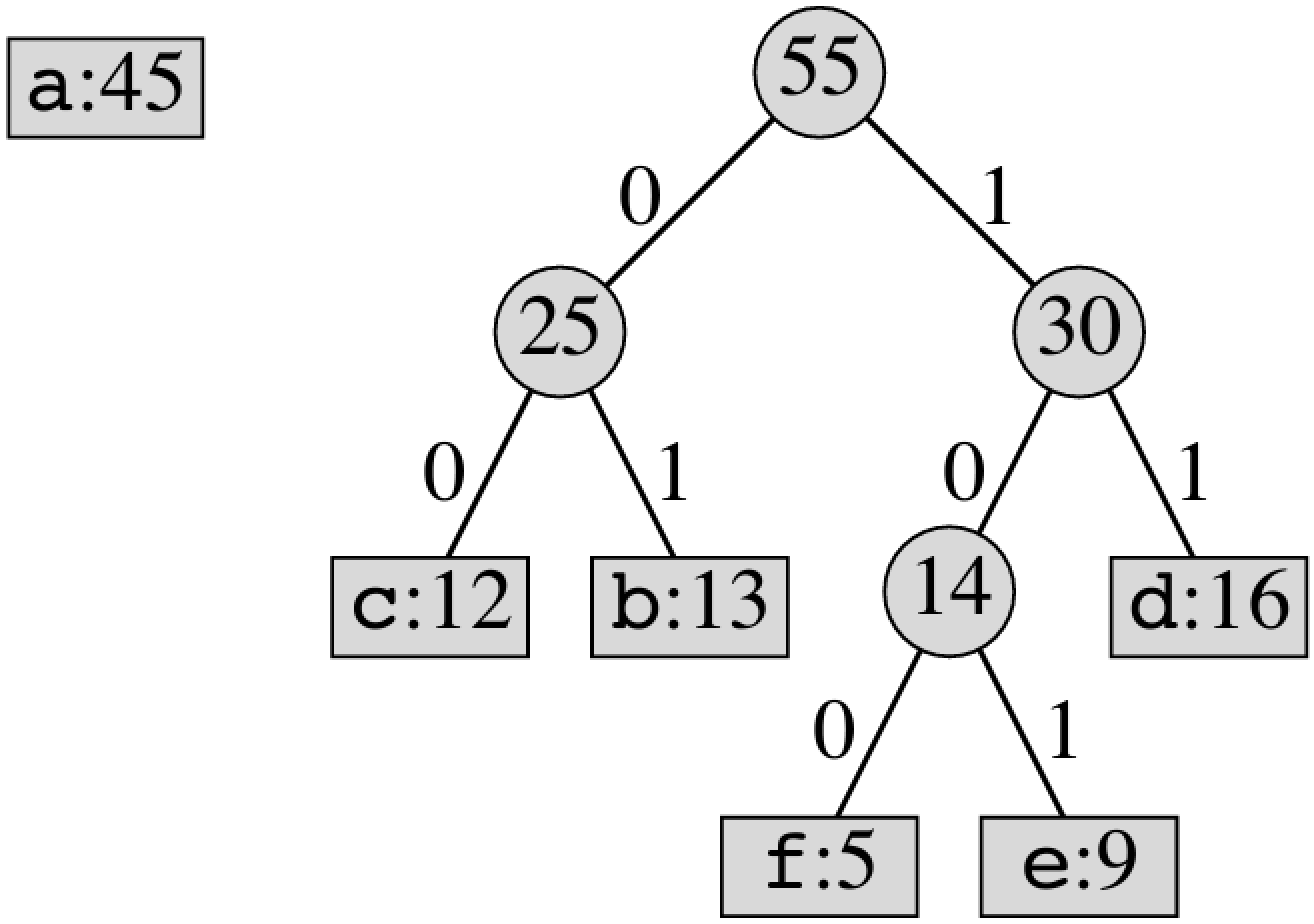

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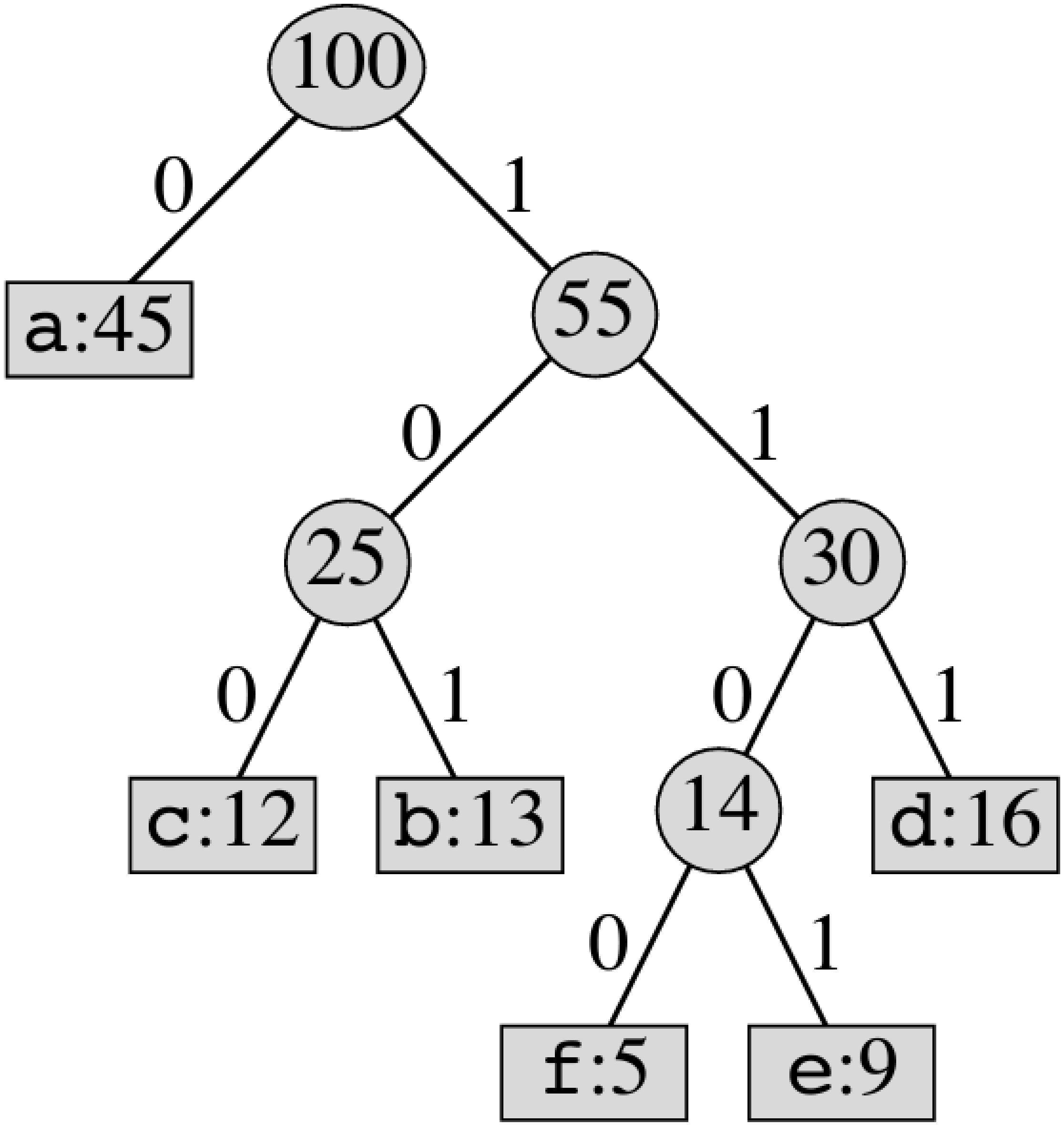

Huffman Codes

- Details of optimal substructure and greedy choice property in the text book

Book Readings and Credits

- Book Readings:
  - 16.1 – 16.3
- Credits to:
  - Prof. Ahmed Eldawy notes