CS141: Intermediate Data Structures and Algorithms

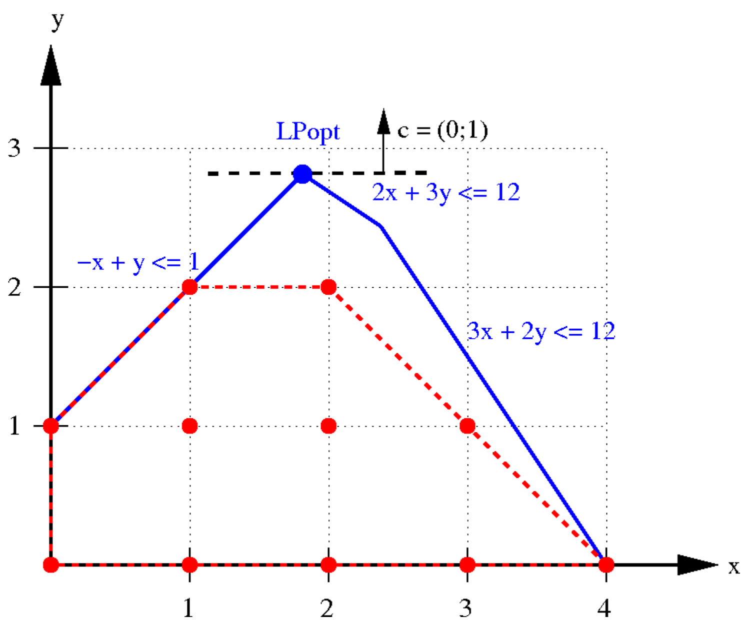
Dynamic Programming

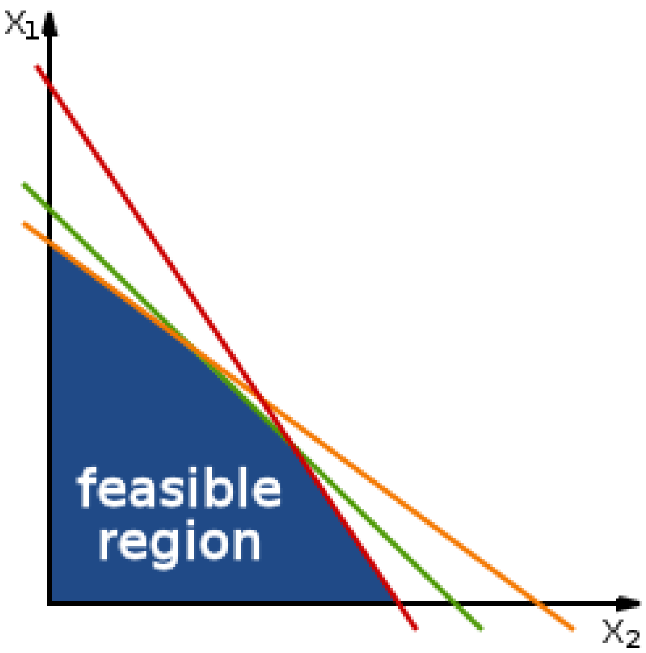
Amr Magdy

Programming?

- In this context, programming is a tabular method
  - Storing previously calculated results in a table, and look it up later

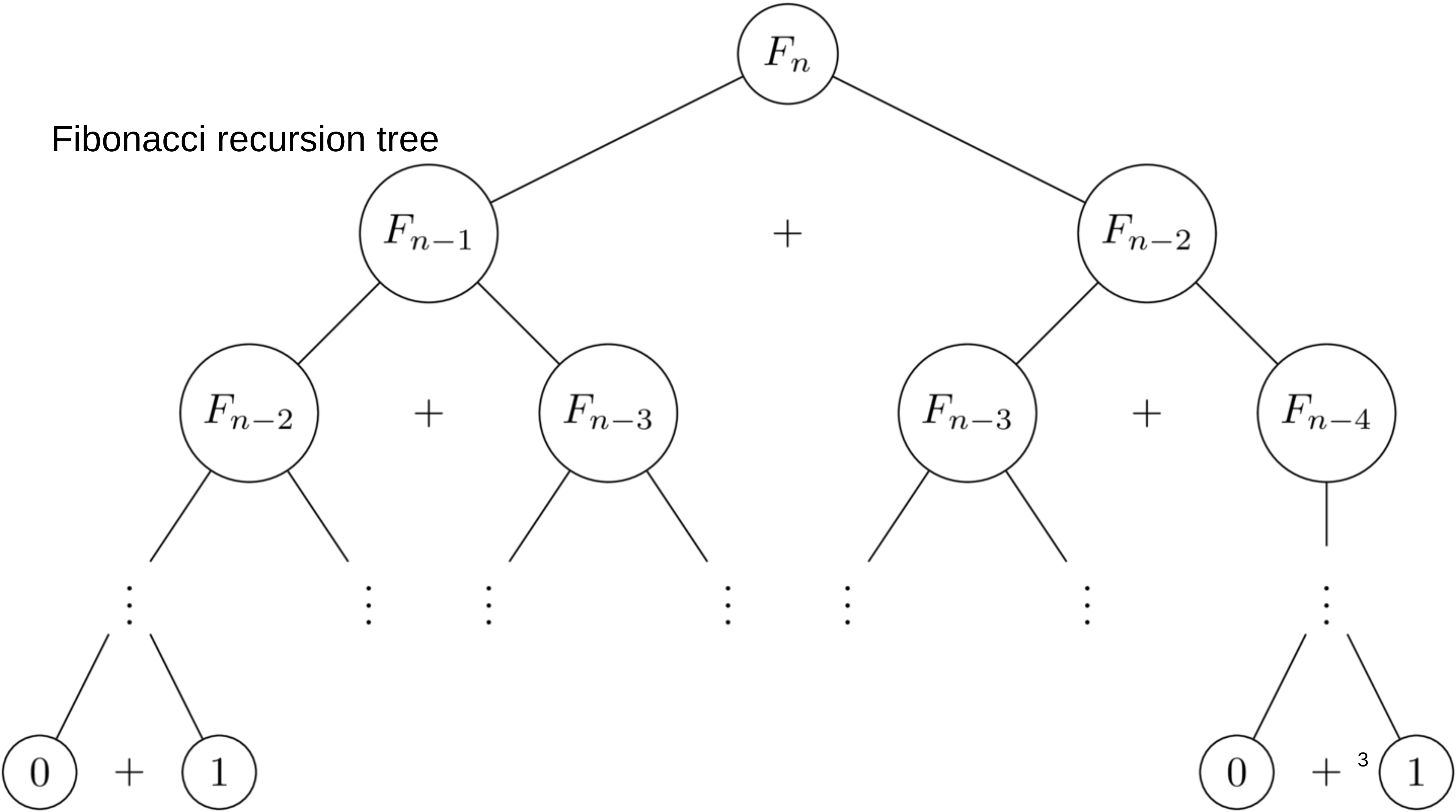
- Other examples:
  - Linear programing
  - Integer programming





Main idea

Fibonacci recursion tree



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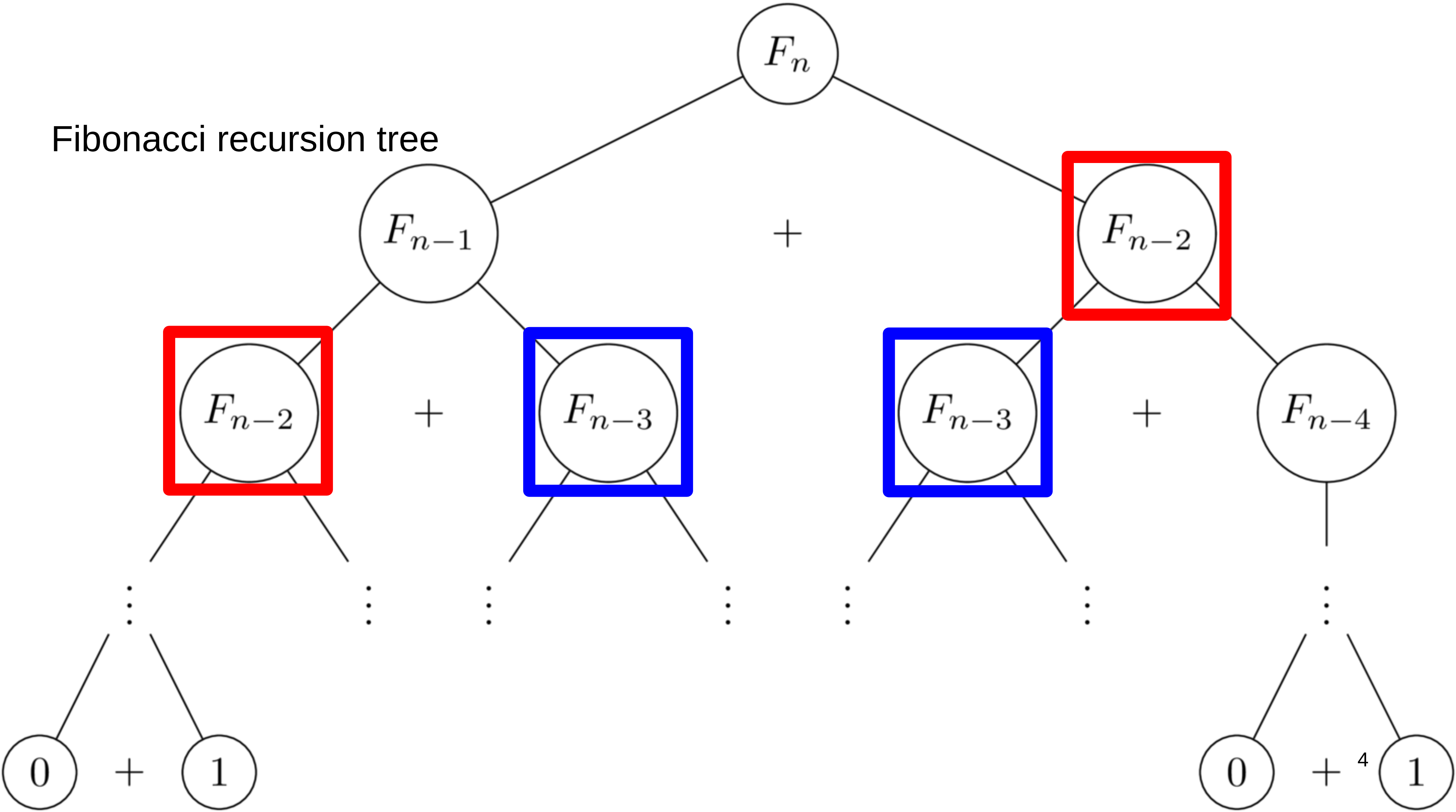

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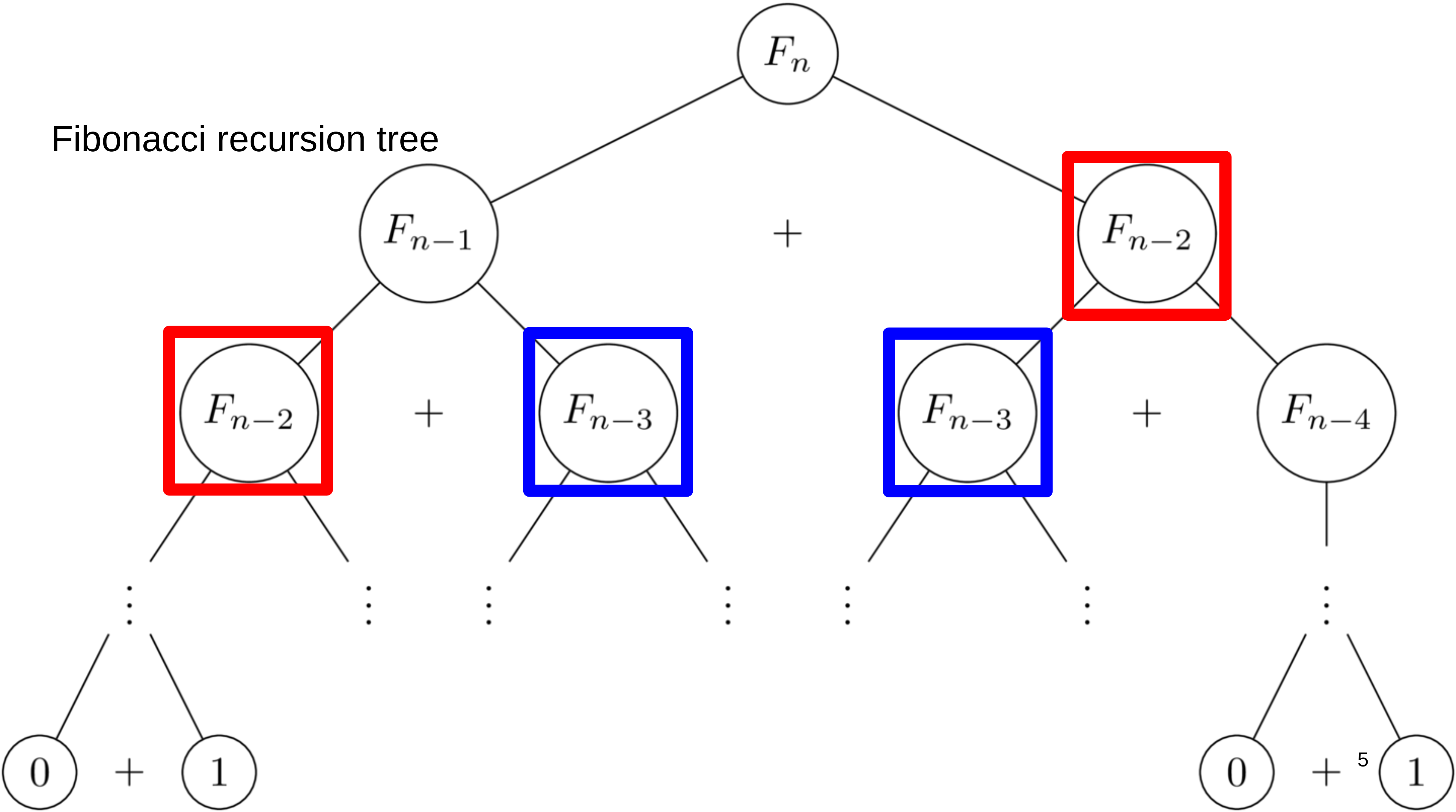
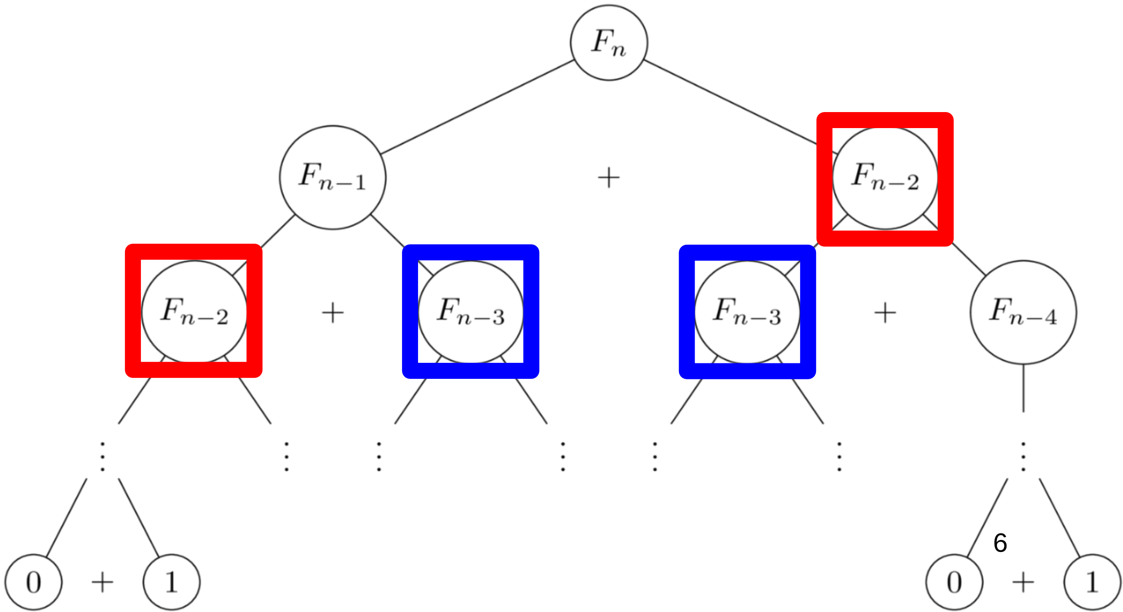
- Do not repeat same work, store the result and look it up later

Fibonacci recursion tree



Main idea: DP vs Divide & Conquer

- Do not repeat same work, store the result and look it up later
- Is MergeSort(A, 1, n/2) and MergeSort(A, n/2, n) the same?
- Is Fib(n-2) and Fib(n-2) the same?



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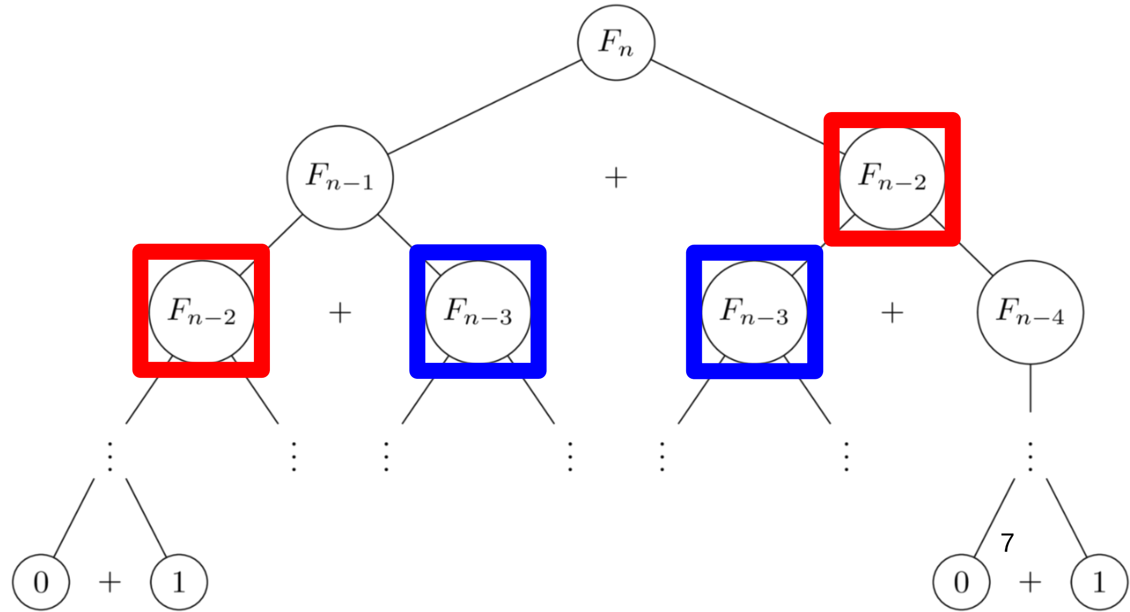

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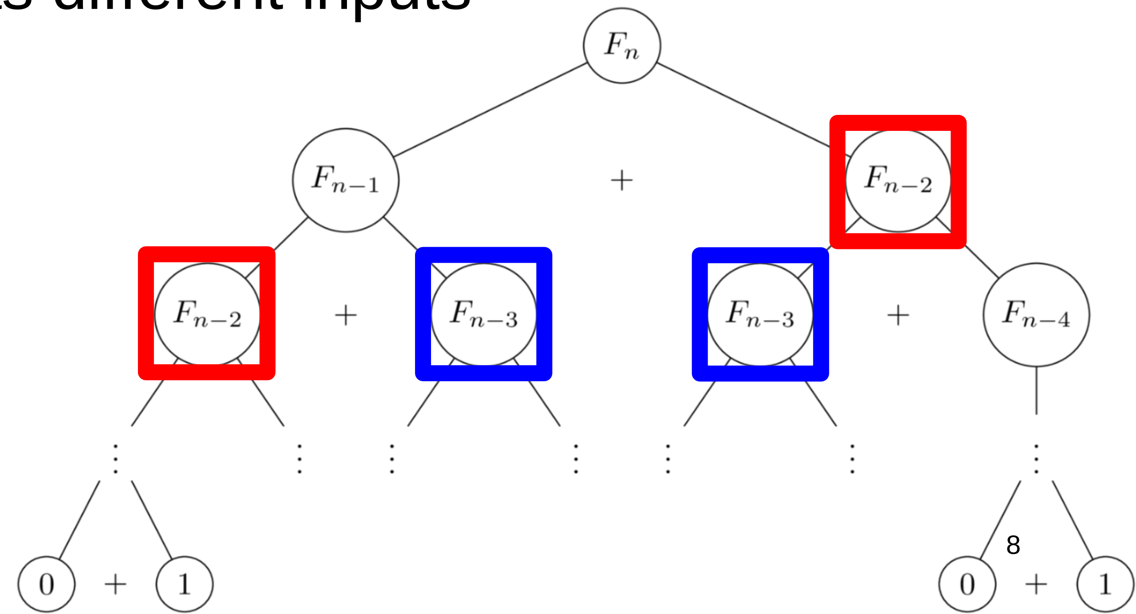
- Same function + same input → same output (DP)
- DC same function has different inputs



Rod Cutting Problem

Rod Cutting Problem

Name: Student ID:

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

The table gives the price pi for a rod of metal of length i, for example, a piece of length 8 is sold for $20. Given a rod of length 6 as shown in the picture, cut this rod to sell it for maximum price. For example, if you cut this rod into two pieces, each of length 3, you can sell it total for $8+$8=$16.

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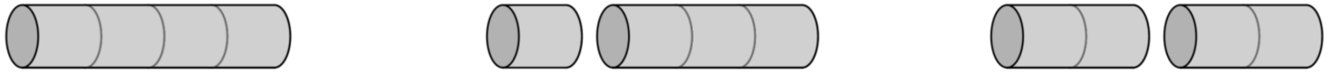

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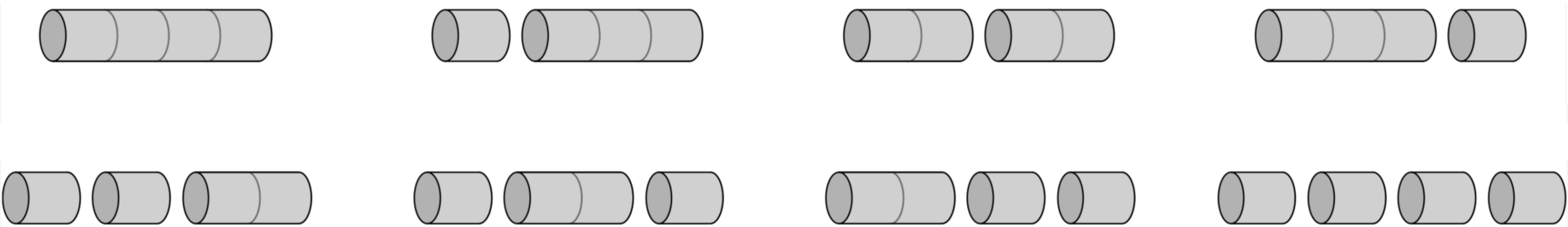

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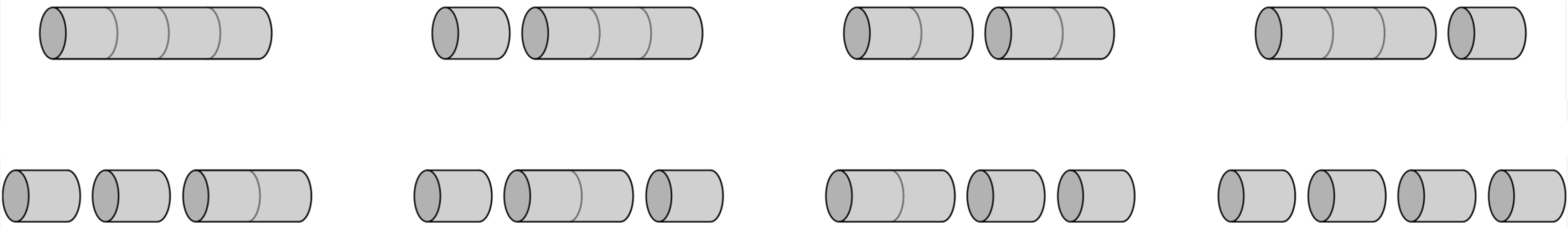

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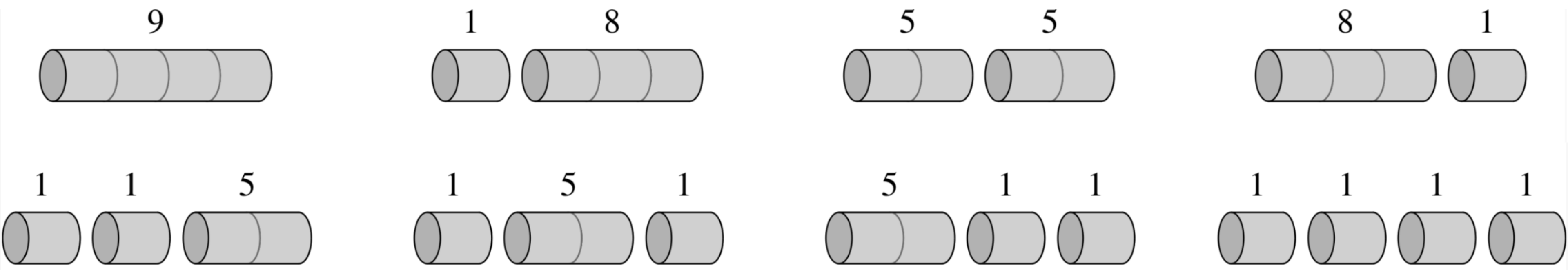

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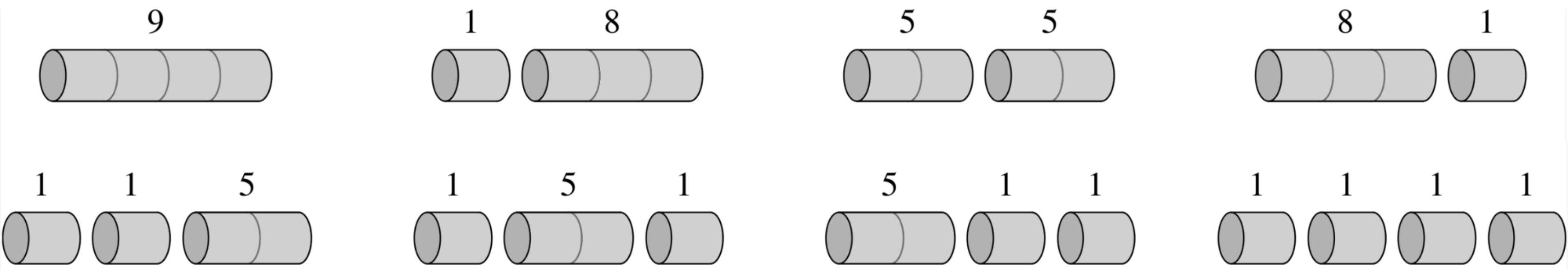

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- Given a rod of length n and prices p_i, find the cutting strategy that makes the maximum revenue
  - In the example: (2+2) cutting makes r=5+5=10

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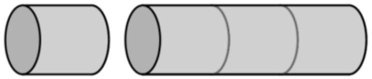
divide



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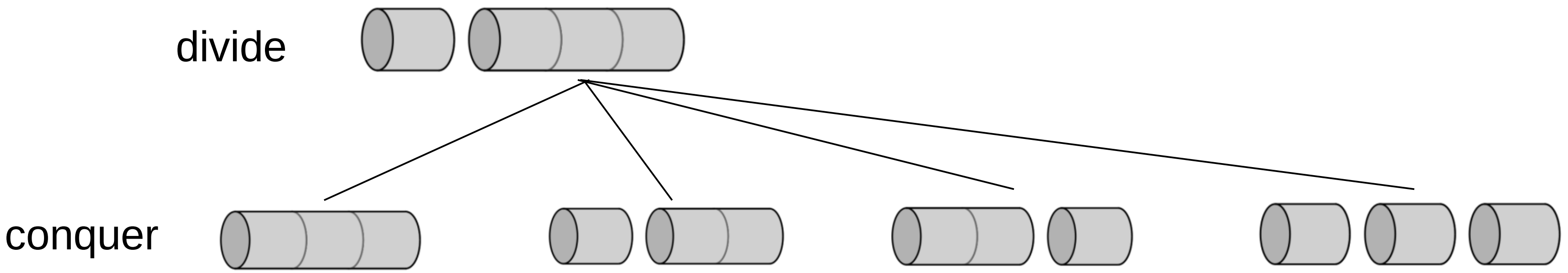

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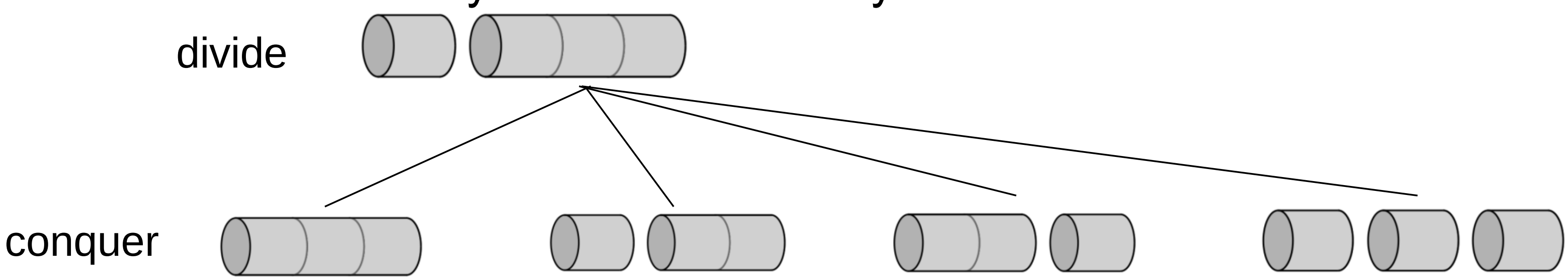

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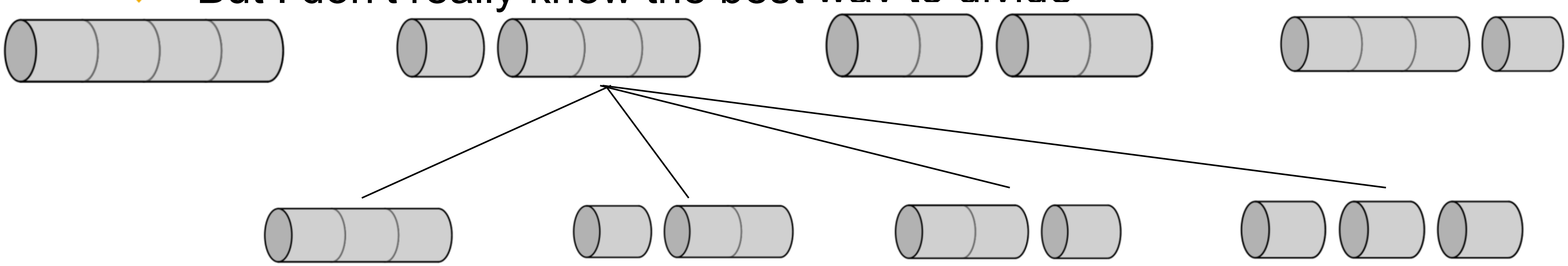

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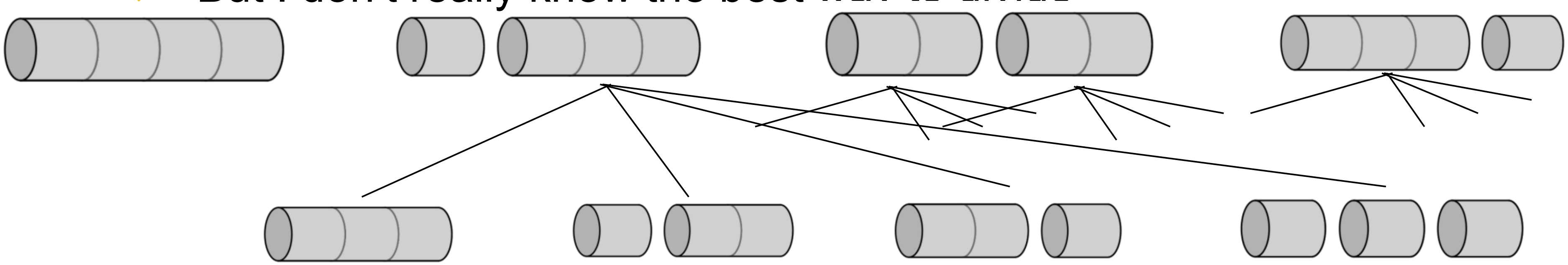

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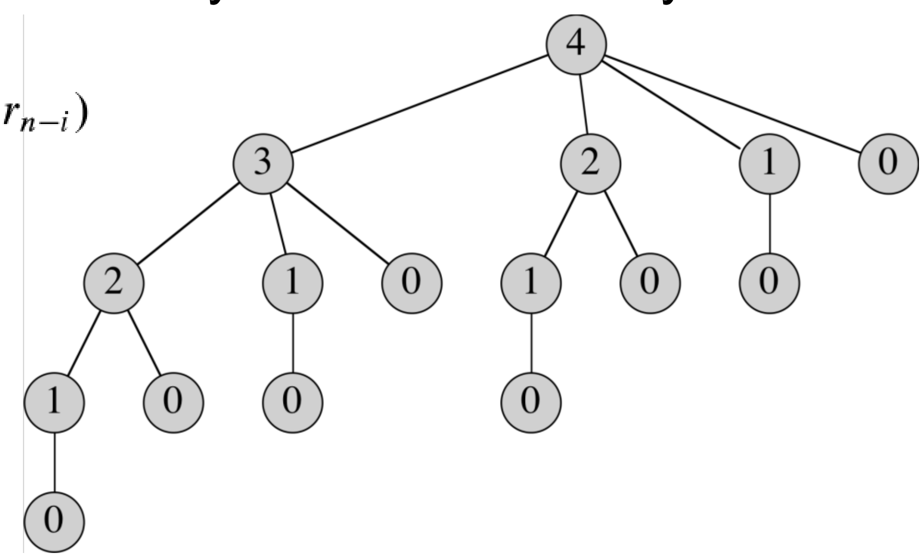

Rod Cutting Problem

- Recursive top-down algorithm

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CUT-ROD(p, n)
if n == 0
    return 0
q = -∞
for i = 1 to n
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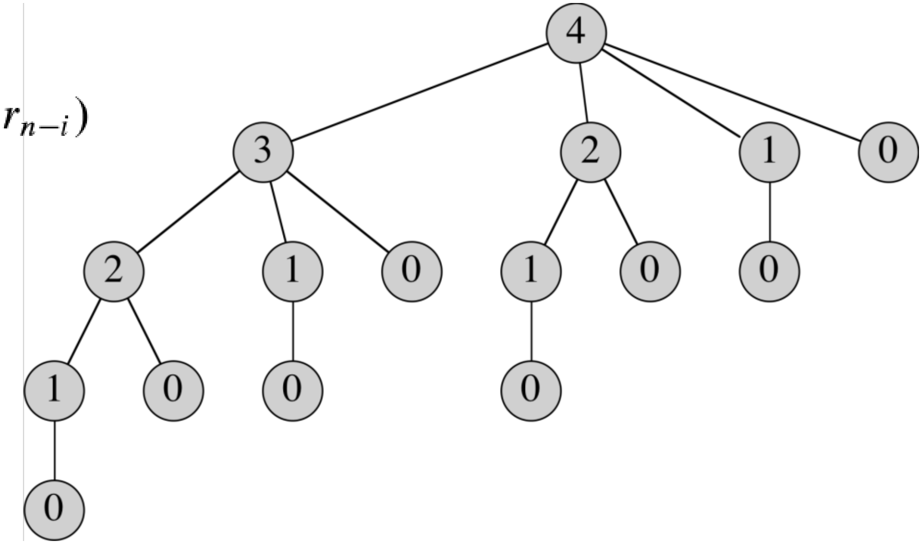

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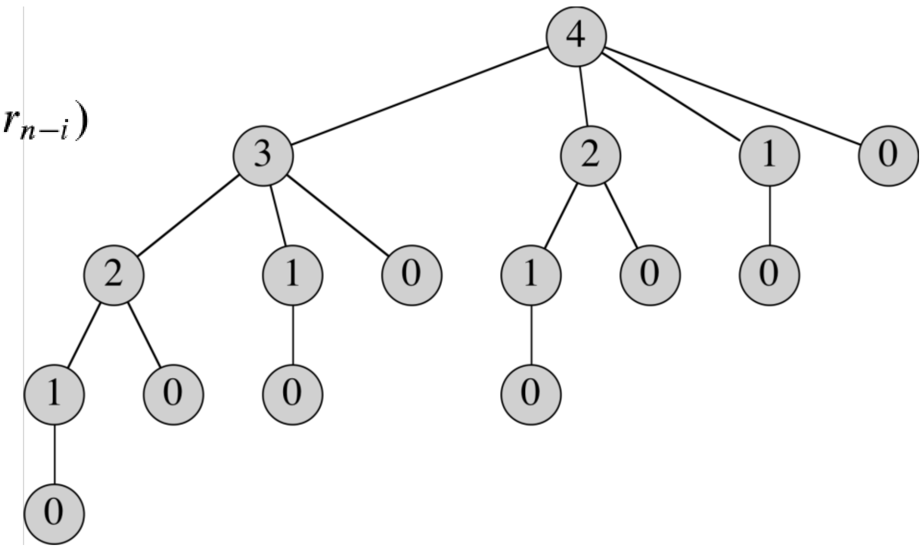

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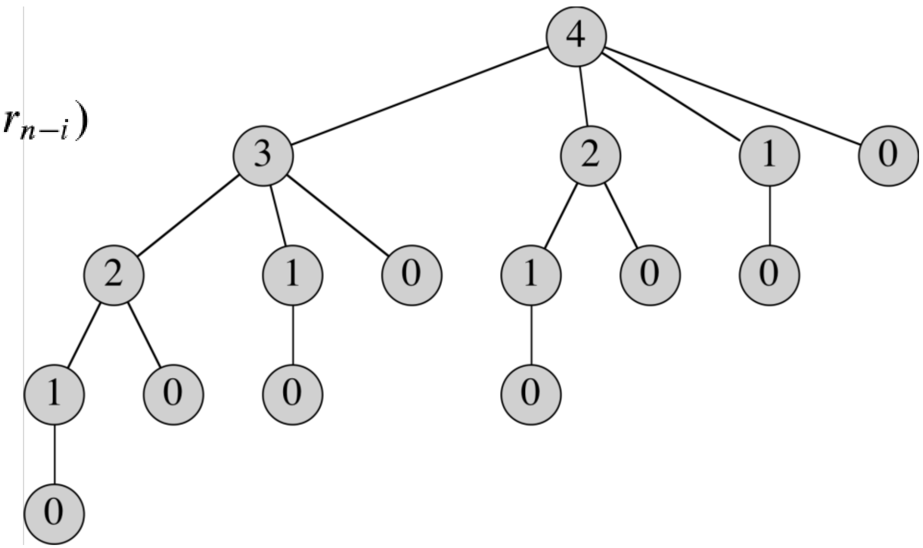

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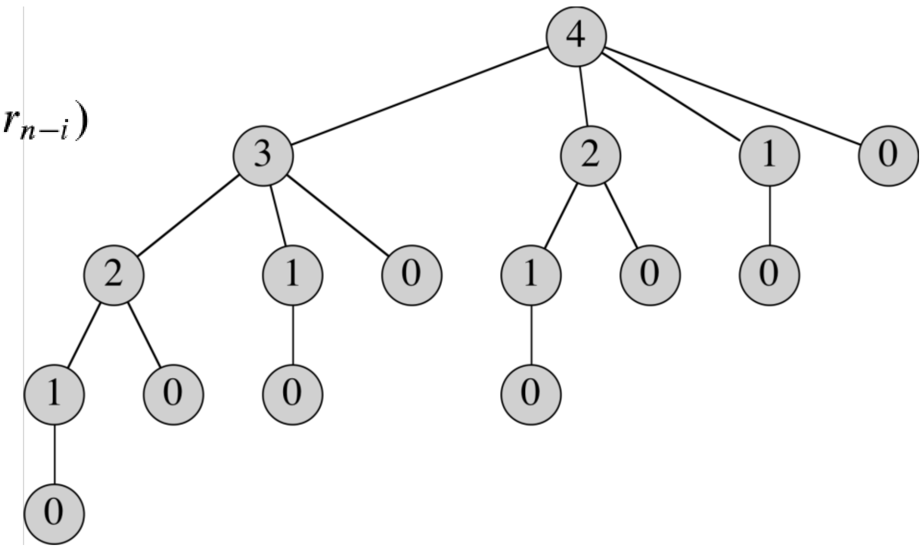

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$$T(n) = 2^n$$

(Prove by induction)

Rod Cutting Recursive Complexity

- Find the complexity of $\mathrm{T(n)} = 1 + \sum_{j=0}^{n-1} T(j)$
- Proof by induction:
  - Assume the solution is some function X(n)
  - Show that X(n) is true for the smallest n (the base case), e.g., n=0
  - Prove that X(n+1) is a solution for T(n+1) given X(n)
  - You are done
- Given $\mathrm{T(n)} = 1 + \sum_{j=0}^{n-1} T(j)$
- Assume $\mathrm{T(n)} = 2^n$
- $\mathrm{T(0)} = 1 + \sum_{j=0}^{-1} T(j) = 1 = 2^0$ (base case)
- $\mathrm{T(n+1)} = 1 + \sum_{j=0}^{n} T(j) = 1 + \sum_{j=0}^{n-1} T(j) + T(n) = T(n) + T(n) = 2T(n) = 2 * 2^n = 2^{n+1}$
- Then, $\mathrm{T(n)} = 2^n$

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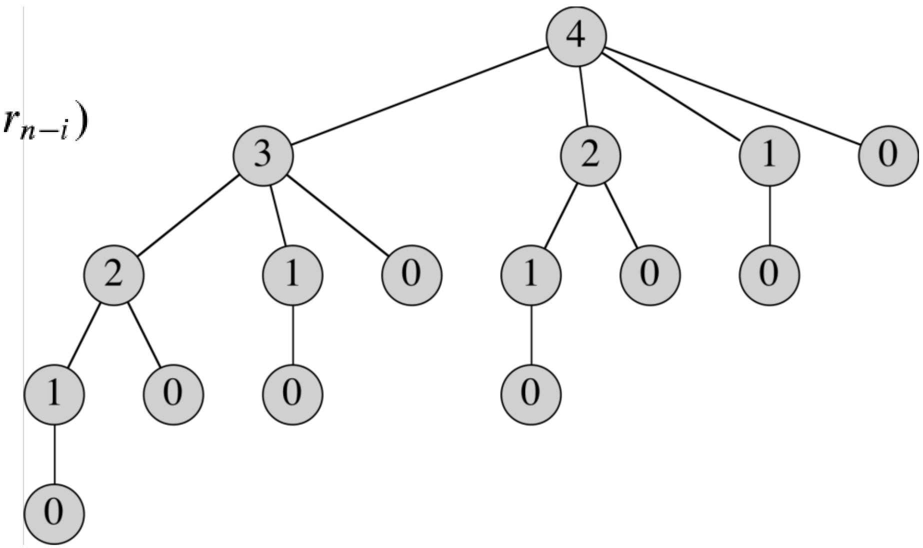

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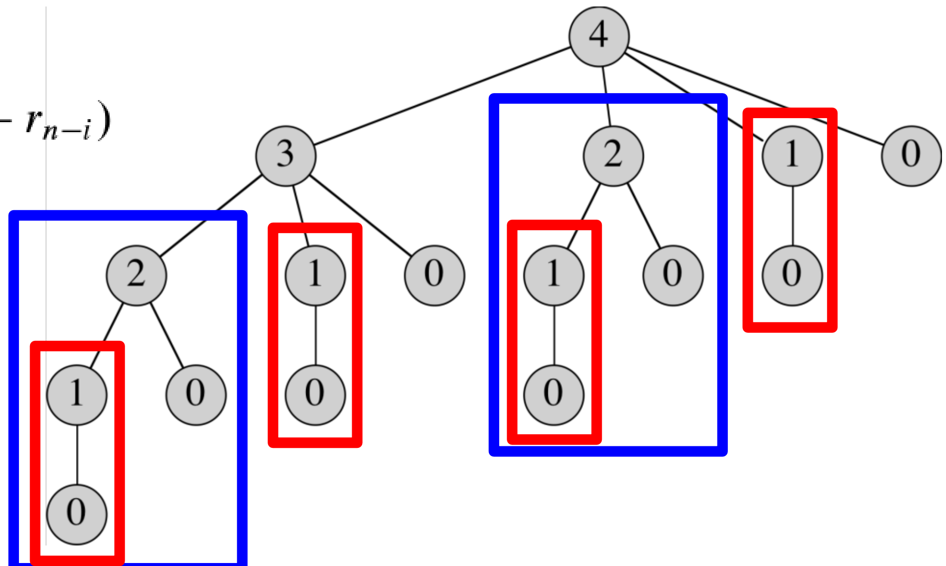

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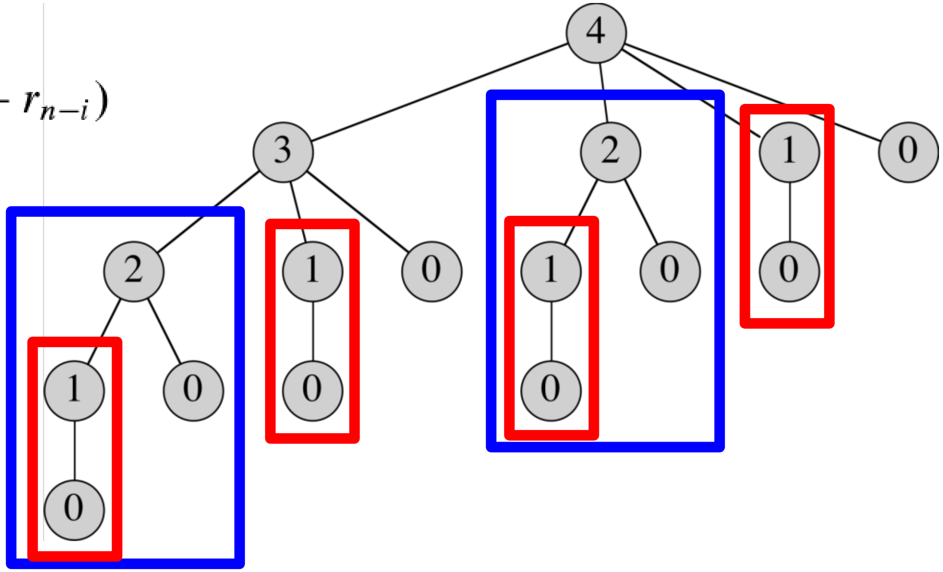

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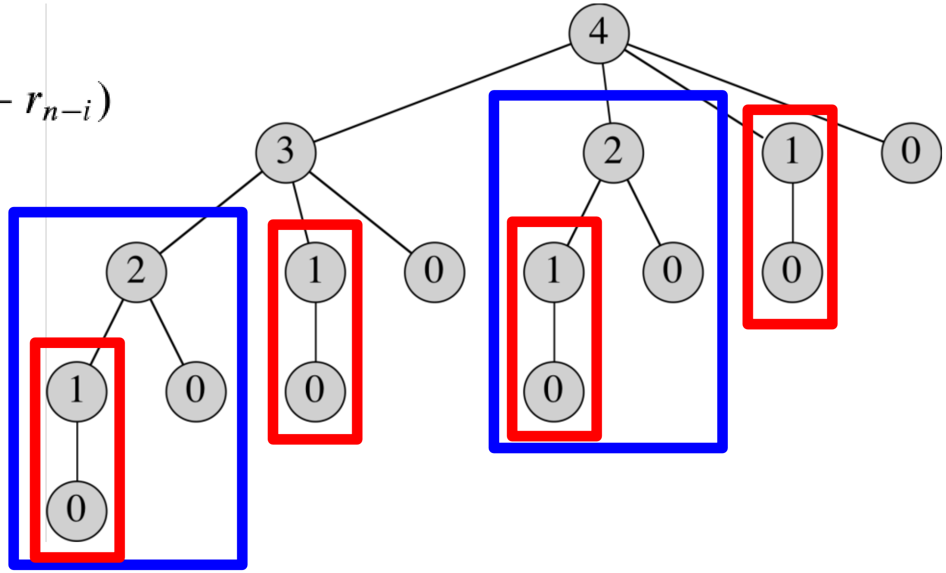

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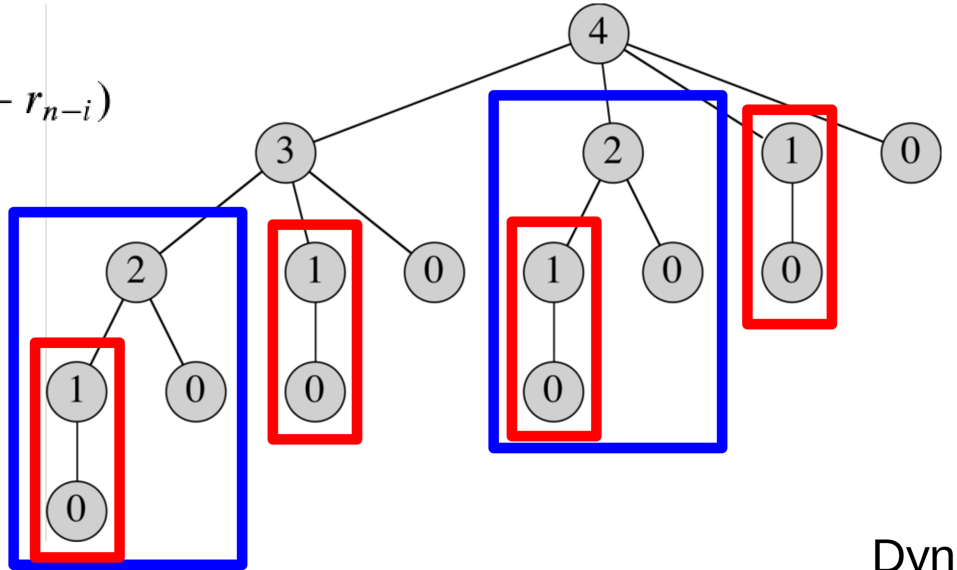

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Rod Cutting Problem

- Recursive top-down dynamic programming algorithm

```
MEMOIZED-CUT-ROD(p, n)
let r[0..n] be a new array
for i = 0 to n
    r[i] = −∞
return MEMOIZED-CUT-ROD-AUX(p, n, r)
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MEMOIZED-CUT-ROD-AUX(p, n, r)
if r[n] ≥ 0
    return r[n]
if n == 0
    q = 0
else q = −∞
    for i = 1 to n
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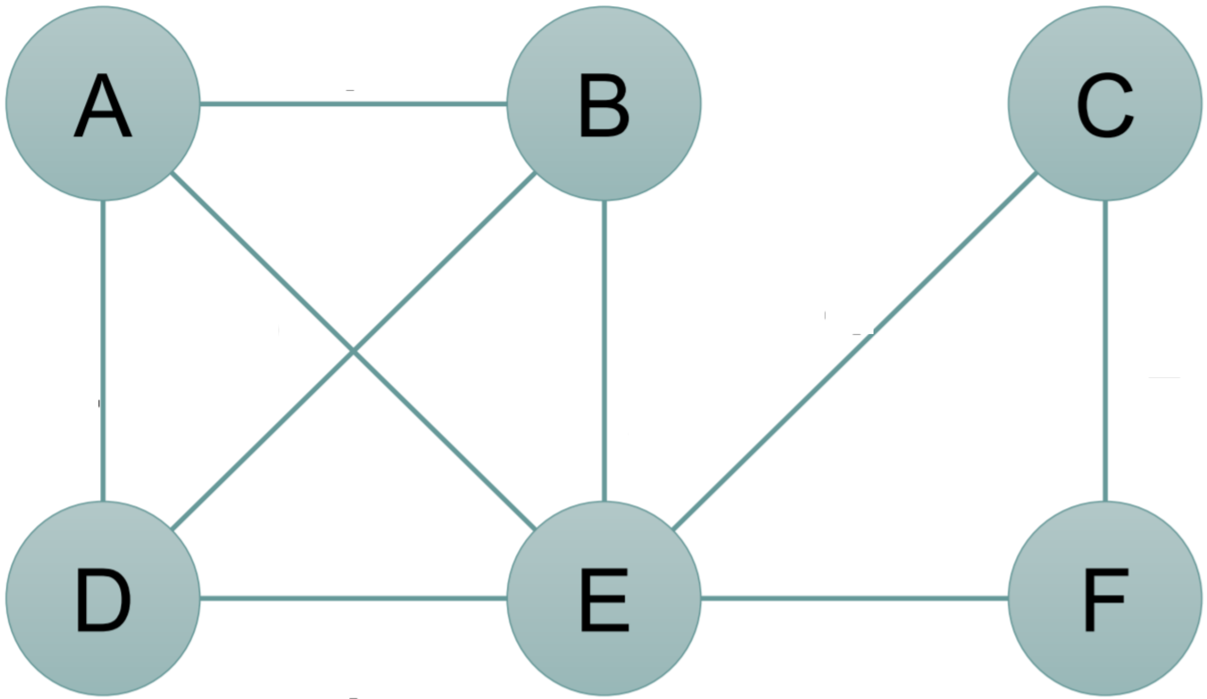
Elements of a Dynamic Programming Problem

- Optimal substructure
  - Optimal solution of a larger problem comes from optimal solutions of smaller problems
- Subproblem overlapping
  - Same exact sub-problems are solved again and again

Optimal Substructure

- Longest path from A to F LP(A,F) includes node B
  - But it does not include LP(A,B) and LP(B,F)
  - i.e., optimal solutions for the subproblems A→B, and B→F cannot be combined to find an optimal solution for A → F



Dynamic Programming vs. D&C

- How different?

Dynamic Programming vs. D&C

- How different?
  - No subproblem overlapping
    - Each subproblem with distinct input is a new problem
  - Not necessarily optimization problems, i.e., no objective function

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EXTENDED-BOTTOM-UP-CUT-ROD(p, n)

```
let r[0..n] and s[1..n] be new arrays
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        if q < p[i] + r[j - i]
            q = p[i] + r[j - i]
            s[j] = i
    r[j] = q
return r and s
```

Reconstructing Solution

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PRINT-CUT-ROD-SOLUTION(p, n)

```
(r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
while n > 0
    print s[n]
    n = n - s[n]
```

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$		1	2	3	2	2	6	1	2	3	10

  - Let's trace examples

Matrix Chain Multiplication

- How to multiply a chain of four matrices $A_1 A_2 A_3 A_4$?

Matrix Chain Multiplication

- How to multiply a chain of four matrices $A_1 A_2 A_3 A_4$?

$$(A_1(A_2(A_3 A_4)))$$
$$(A_1((A_2 A_3)A_4))$$
$$((A_1 A_2)(A_3 A_4))$$
$$((A_1(A_2 A_3))A_4)$$
$$(((A_1 A_2)A_3)A_4)$$

Matrix Chain Multiplication

- How to multiply a chain of four matrices $A_1 A_2 A_3 A_4$?

$$(A_1(A_2(A_3A_4)))$$
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- Does it really make a difference?

Matrix Chain Multiplication

- How to multiply a chain of four matrices $A_1 A_2 A_3 A_4$?

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$$(A_1((A_2A_3)A_4))$$
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$$((A_1(A_2A_3))A_4)$$
$$(((A_1A_2)A_3)A_4)$$

- Does it really make a difference?
- # of multiplications:

A.rows*B.cols*A.cols

MATRIX-MULTIPLY(A, B)

```
if A.columns ≠ B.rows
    error "incompatible dimensions"
else let C be a new A.rows × B.columns matrix
    for i = 1 to A.rows
        for j = 1 to B.columns
            c_ij = 0
            for k = 1 to A.columns
                c_ij = c_ij + a_ik · b_kj
    return C
```

Matrix Chain Multiplication

- Does it really make a difference?
- # of multiplications:

A.rows*B.cols*A.cols

- Example:

A1*A2*A3

Dimensions:

10x100x5x50

```
MATRIX-MULTIPLY(A, B)
if A.columns ≠ B.rows
    error "incompatible dimensions"
else let C be a new A.rows × B.columns matrix
    for i = 1 to A.rows
        for j = 1 to B.columns
            c_ij = 0
            for k = 1 to A.columns
                c_ij = c_ij + a_ik · b_kj
    return C
```

- # of multiplications in ((A1*A2)*A3)=10*100*5+10*5*50=7.5K
- # of multiplications in (A1*(A2*A3))=100*5*50+10*100*50=75K

Matrix Chain Multiplication

- Given n matrices A_1 A_2 … A_n of dimensions p_0 p_1 … p_n, find the optimal parentheses to multiply the matrix chain

Matrix Chain Multiplication

- Given n matrices $A_1\ A_2\ \dots\ A_n$ of dimensions $p_0\ p_1\ \dots\ p_n$, find the optimal parentheses to multiply the matrix chain
- $A_1\ A_2\ A_3\ A_4\ A_5\ \dots\ A_n$

Matrix Chain Multiplication

- Given n matrices A_1 A_2 … A_n of dimensions p_0 p_1 … p_n, find the optimal parentheses to multiply the matrix chain
- $A_1 \, A_2 \, A_3 \, A_4 \, A_5 \ldots A_n$
- $(A_1 \, A_2 \, A_3)(A_4 \, A_5 \ldots A_n)$

Matrix Chain Multiplication

- Given n matrices $A_1\ A_2 \dots A_n$ of dimensions $p_0\ p_1 \dots p_n$, find the optimal parentheses to multiply the matrix chain
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- $(A_1\ A_2\ A_3)(A_4\ A_5 \dots A_n)$
- Sub-chains C1 = $(A_1\ A_2\ A_3)$, C2 = $(A_4\ A_5 \dots A_n)$

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- Total Cost C = cost(C1)+cost(C2)+$p_0 p_3 p_n$

Matrix Chain Multiplication

- Given n matrices $A_1\ A_2 \ldots A_n$ of dimensions $p_0\ p_1 \ldots p_n$, find the optimal parentheses to multiply the matrix chain
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- Then, if cost(C1) and cost(C2) are minimal (i.e., optimal), then C is optimal (optimal substructure holds)

Matrix Chain Multiplication

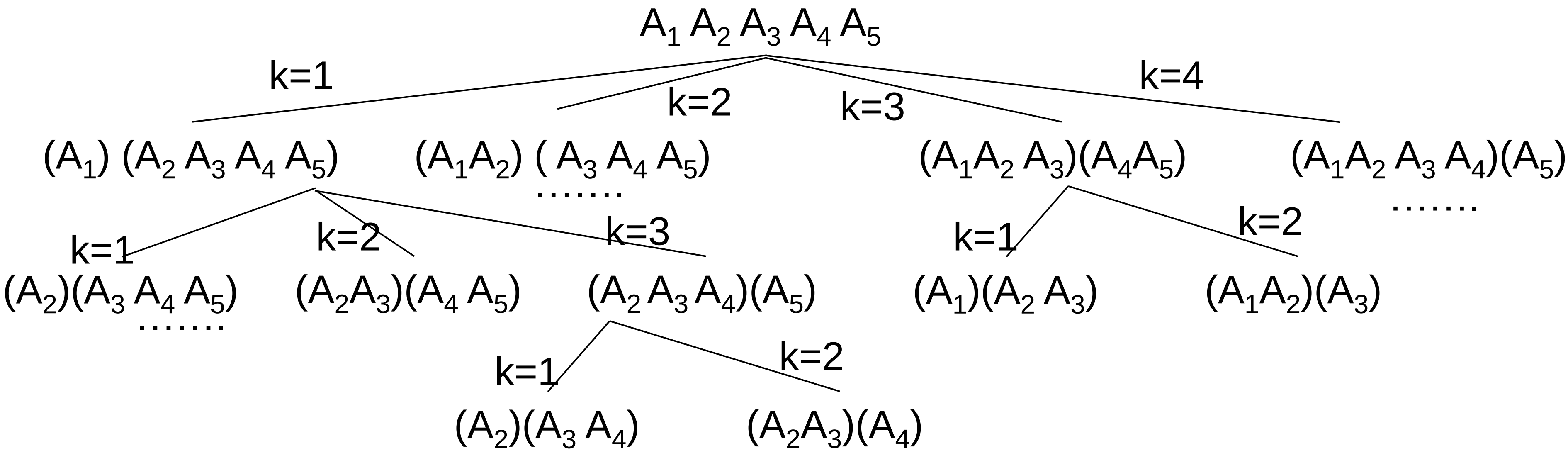
- Given n matrices A_1 A_2 … A_n of dimensions p_0 p_1 … p_n, find the optimal parentheses to multiply the matrix chain
- A_1 A_2 A_3 A_4 A_5 … A_n
- (A_1 A_2 A_3)(A_4 A_5 … A_n)
- Sub-chains C1 = (A_1 A_2 A_3), C2 = (A_4 A_5 … A_n)
- Total Cost C = cost(C1)+cost(C2)+$p_0p_3p_n$
- Then, if cost(C1) and cost(C2) are minimal (i.e., optimal), then C is optimal (optimal substructure holds)
- Proof by contradiction:
  - Given C is optimal, are cost(C1)=c1 and cost(C2)=c2 optimal?
  - Assume c1 is NOT optimal, then ∃ an optimal solution of cost c1' < c1
  - Then c1'+c2+p < c1+c2+p → C' < C
  - Then C is not optimal → contradiction!
  - Then C1 has to be optimal → optimal substructure holds

Matrix Chain Multiplication

- Given n matrices $A_1\ A_2 \dots A_n$ of dimensions $p_0\ p_1 \dots p_n$, find the optimal parentheses to multiply the matrix chain
- $A_1\ A_2\ A_3\ A_4\ A_5 \dots A_n$
- $(A_1\ A_2\ A_3)(A_4\ A_5 \dots A_n)$
- Sub-chains $C1 = (A_1\ A_2\ A_3)$, $C2 = (A_4\ A_5 \dots A_n)$
- Total Cost C = cost(C1)+cost(C2)+$p_0p_3p_n$
- Then, if cost(C1) and cost(C2) are minimal (i.e., optimal), then C is optimal (optimal substructure holds)
- Optimal C1, C2 might be one of different options
  - $C1 = (A_1\ A_2)$, $C2 = (A_3\ A_4\ A_5 \dots A_n)$
  - $C1 = (A_1)$, $C2 = (A_2\ A_3\ A_4\ A_5 \dots A_n)$
  - $C1 = (A_1\ A_2\ A_3\ A_4)$, $C2 = (A_5 \dots A_n)$
  - …….

Matrix Chain Multiplication

- Assume k is length of first sub-chain C1



Matrix Chain Multiplication

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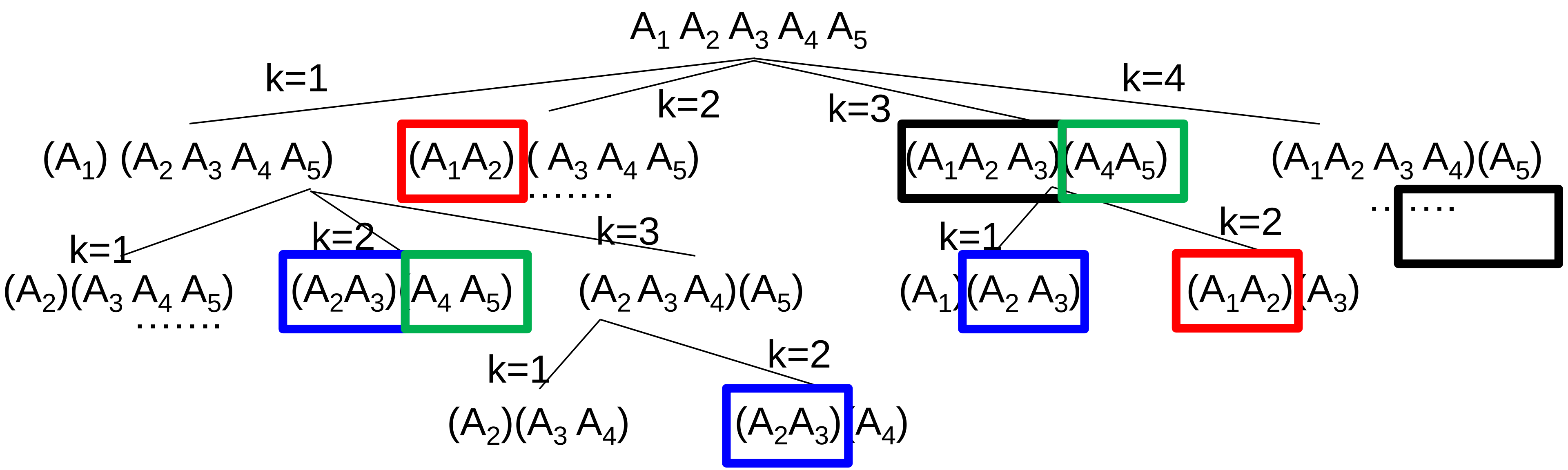

Matrix Chain Multiplication

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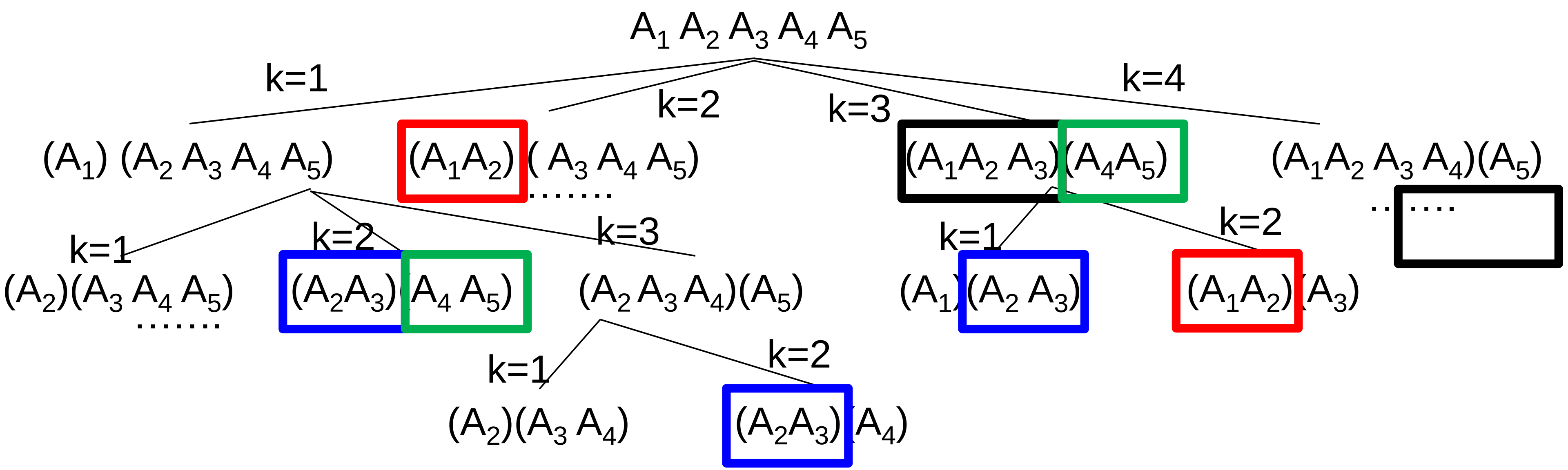
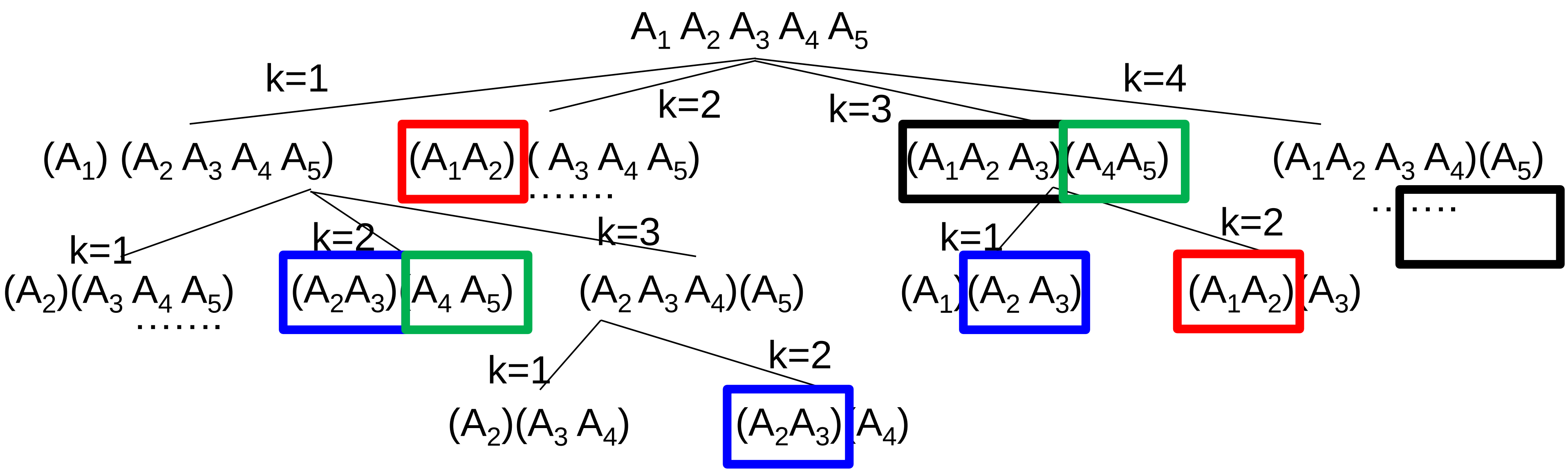


- Obviously, a lot of overlapping subproblems appear

Matrix Chain Multiplication

- Assume k is length of first sub-chain C1



- Obviously, a lot of overlapping subproblems appear
- Optimal substructure + subproblem overlapping = dynamic programming

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
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- Solve all chains of length 2, then 3, then 4, …n

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,

A1: 30x35

A2: 35x15

A3: 15x5

A4: 5x10

A5: 10x20

A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 2:

(A1 A2)

(A2 A3)

(A3 A4)

(A4 A5)

(A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 2:

(A1 A2) = 30*35*15 = 15750

(A2 A3)

(A3 A4)

(A4 A5)

(A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 2:

(A1 A2) = 30*35*15 = 15750

(A2 A3) = 35*15*5 = 2625

(A3 A4)

(A4 A5)

(A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 2:

(A1 A2) = 30*35*15 = 15750

(A2 A3) = 35*15*5 = 2625

(A3 A4) = 15*5*10 = 750

(A4 A5) = 5*10*20 = 1000

(A5 A6) = 10*20*25 = 5000

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 3:

(A1 A2 A3)

(A2 A3 A4)

(A3 A4 A5)

(A4 A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 3:

(A1 A2 A3) = (A1 A2) A3

or A1 (A2 A3)

(A2 A3 A4)

(A3 A4 A5)

(A4 A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 3:

(A1 A2 A3) = (A1 A2) A3 [15750+30*15*5]

or A1 (A2 A3) [30*35*5+2625]

= 7875

(A2 A3 A4)

(A3 A4 A5)

(A4 A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 3:

(A1 A2 A3) = 7875

(A2 A3 A4) = (A2 A3) A4

or A2 (A3 A4)

(A3 A4 A5)

(A4 A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 3:

(A1 A2 A3) = 7875

(A2 A3 A4) = (A2 A3) A4 [2625+35*5*10]

or A2 (A3 A4) [35*15*10+750]

= 4375

(A3 A4 A5)

(A4 A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 3:

(A1 A2 A3) = 7875

(A2 A3 A4) = 4375

(A3 A4 A5) = 2500

(A4 A5 A6) = 3500

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 4:
- (A1 A2 A3 A4) = A1 (A2 A3 A4)

  Or (A1 A2)(A3 A4)

  Or (A1 A2 A3) A4

(A2 A3 A4 A5)

(A3 A4 A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
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A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 4:
- (A1 A2 A3 A4) = A1 (A2 A3 A4)

  Or (A1 A2)(A3 A4)

  Or (A1 A2 A3) A4

  = 9375

(A2 A3 A4 A5)

(A3 A4 A5 A6)

A1: 30x35
A2: 35x15
A3: 15x5
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A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 4:
- (A1 A2 A3 A4) = 9375

(A2 A3 A4 A5) = 7125

(A3 A4 A5 A6) = 5375

A1: 30x35
A2: 35x15
A3: 15x5
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A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 5:
- (A1 A2 A3 A4 A5) = 11875

(A2 A3 A4 A5 A6) = 10500

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n
- Example: n=6,
- Chains of length 6:
- (A1 A2 A3 A4 A5 A6) = 15125

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
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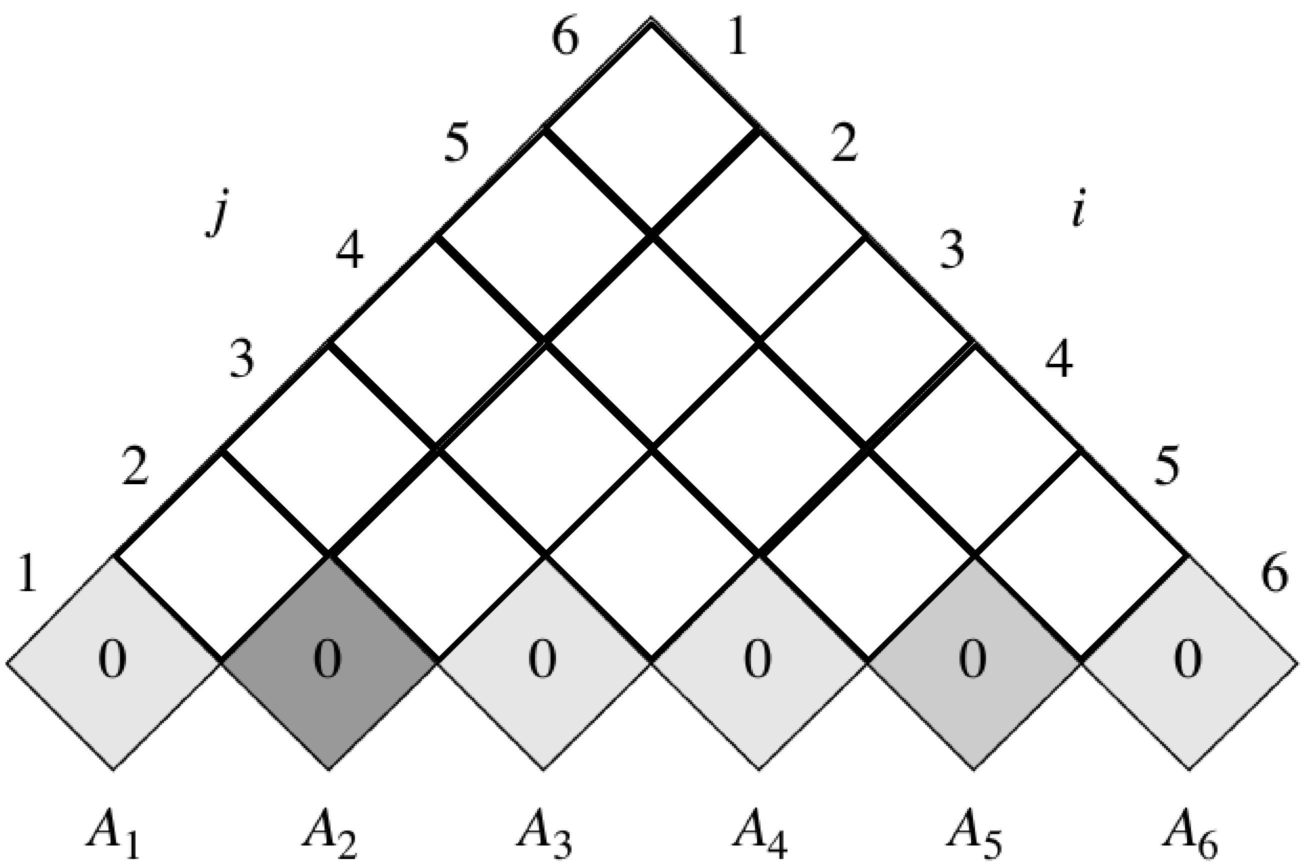
Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n



A1: 30x35
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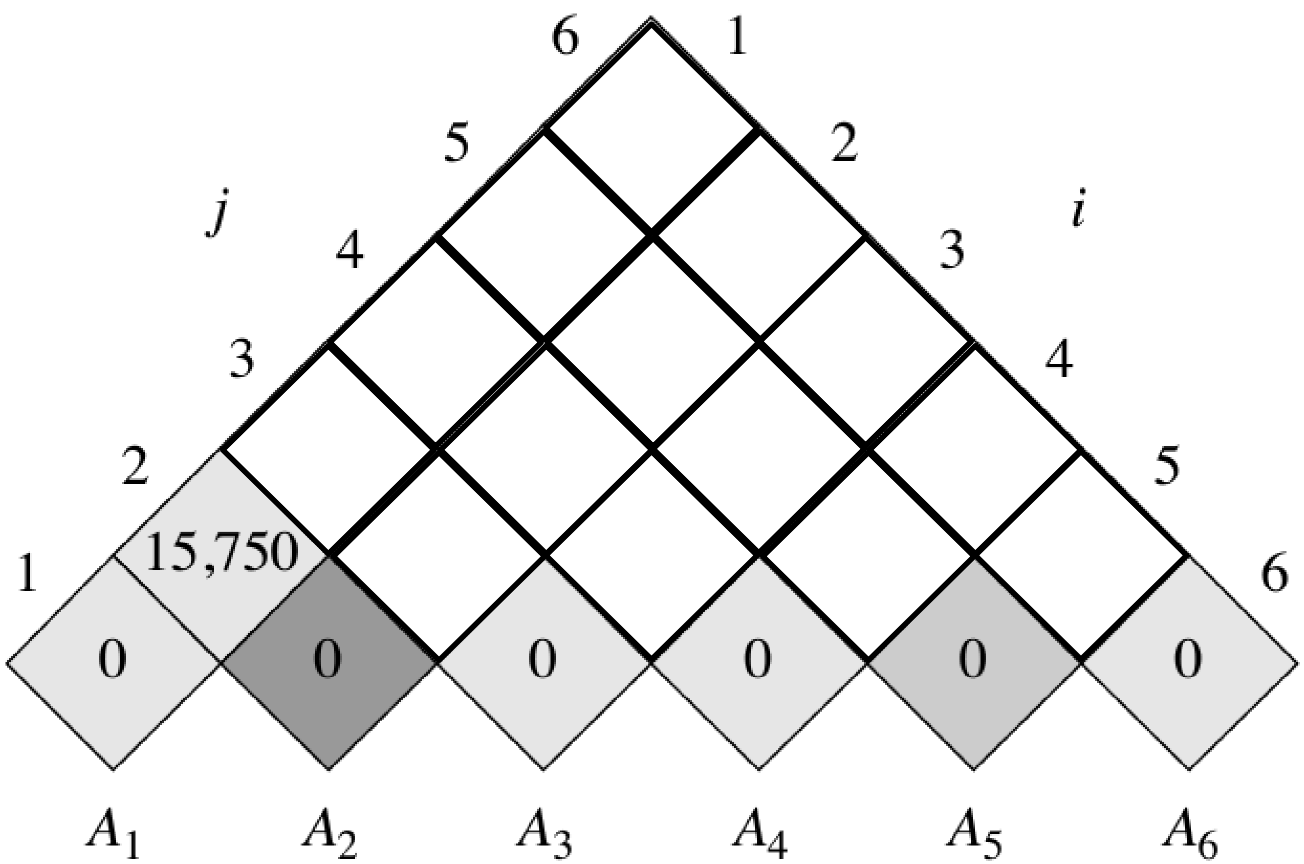
Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n



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Matrix Chain Multiplication: Designing Algorithm

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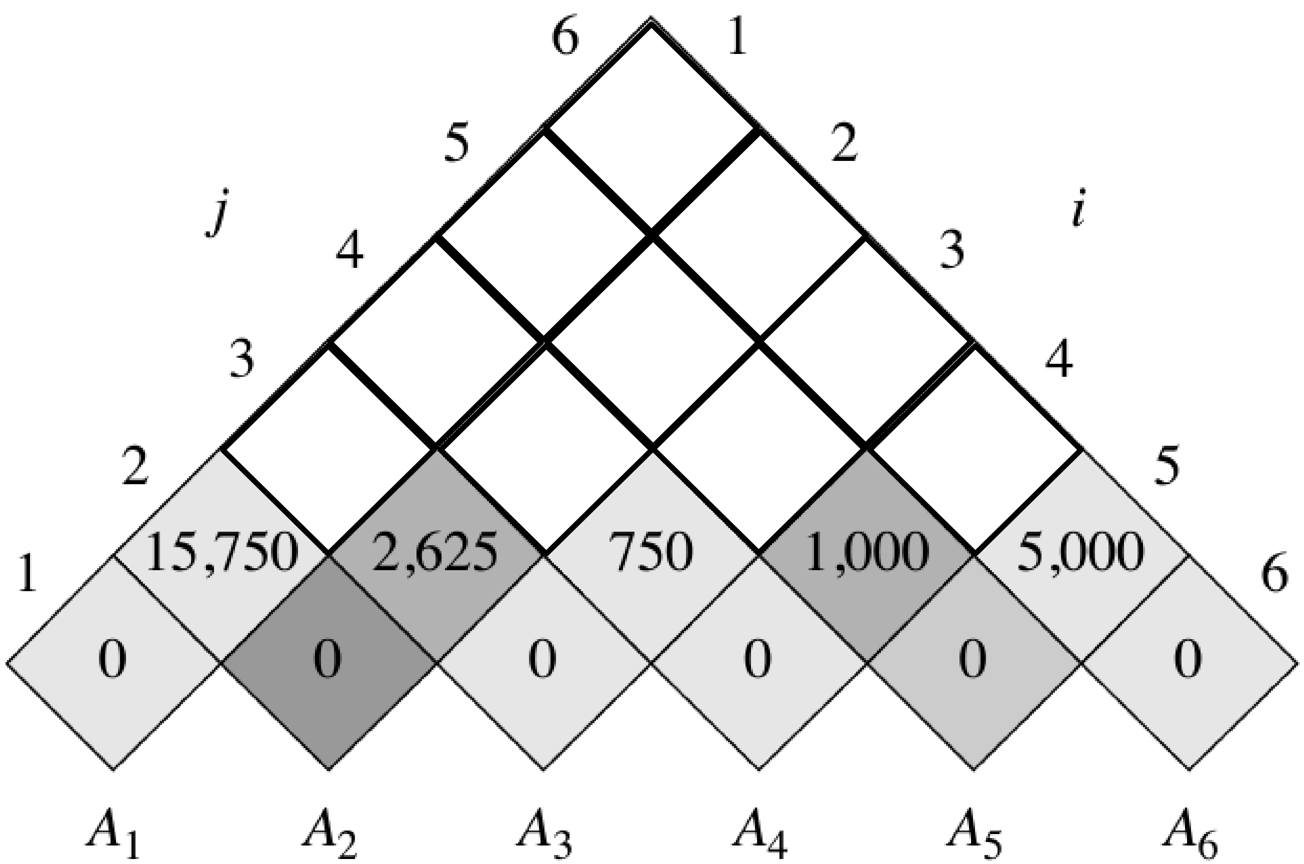

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A2: 35x15
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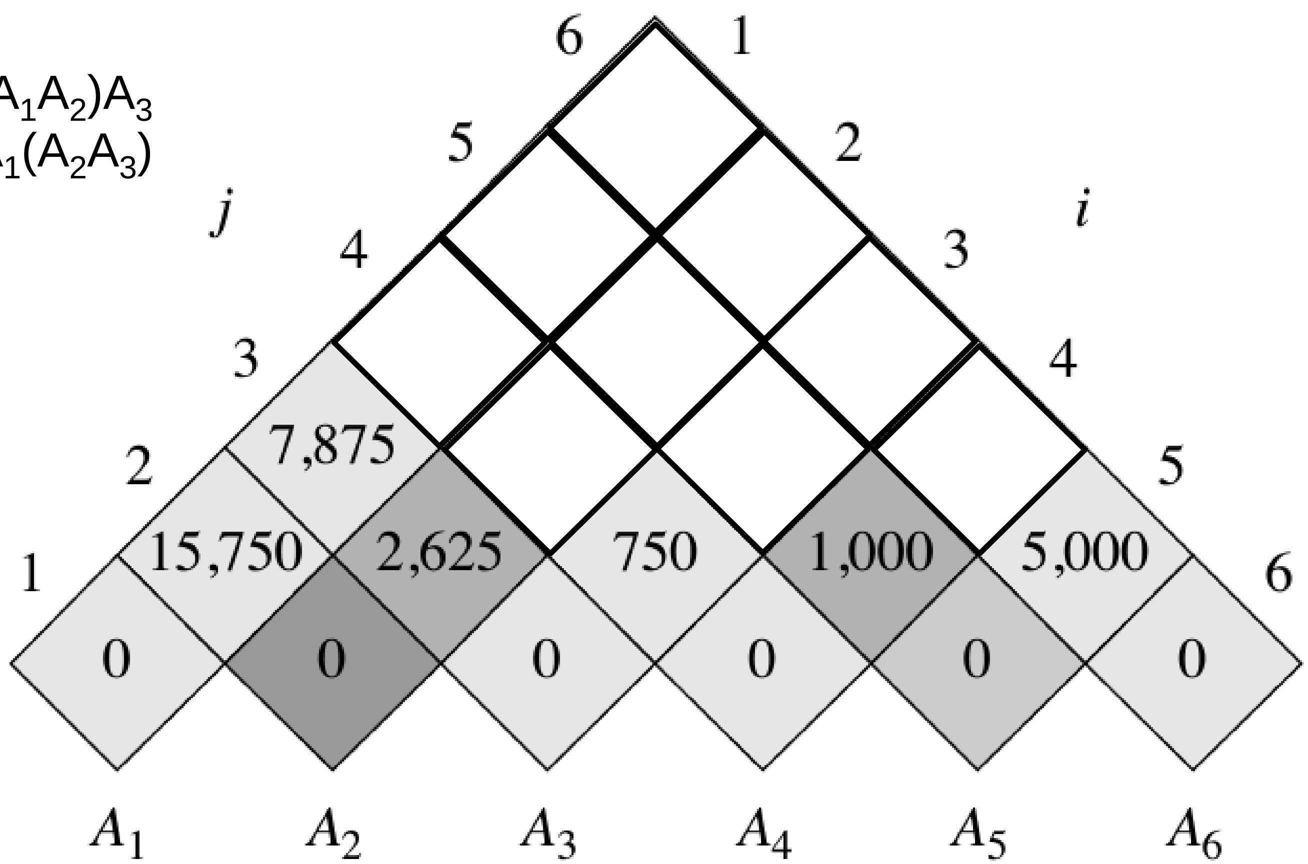
Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n

$A_1A_2A_3 = (A_1A_2)A_3$
Or $A_1(A_2A_3)$



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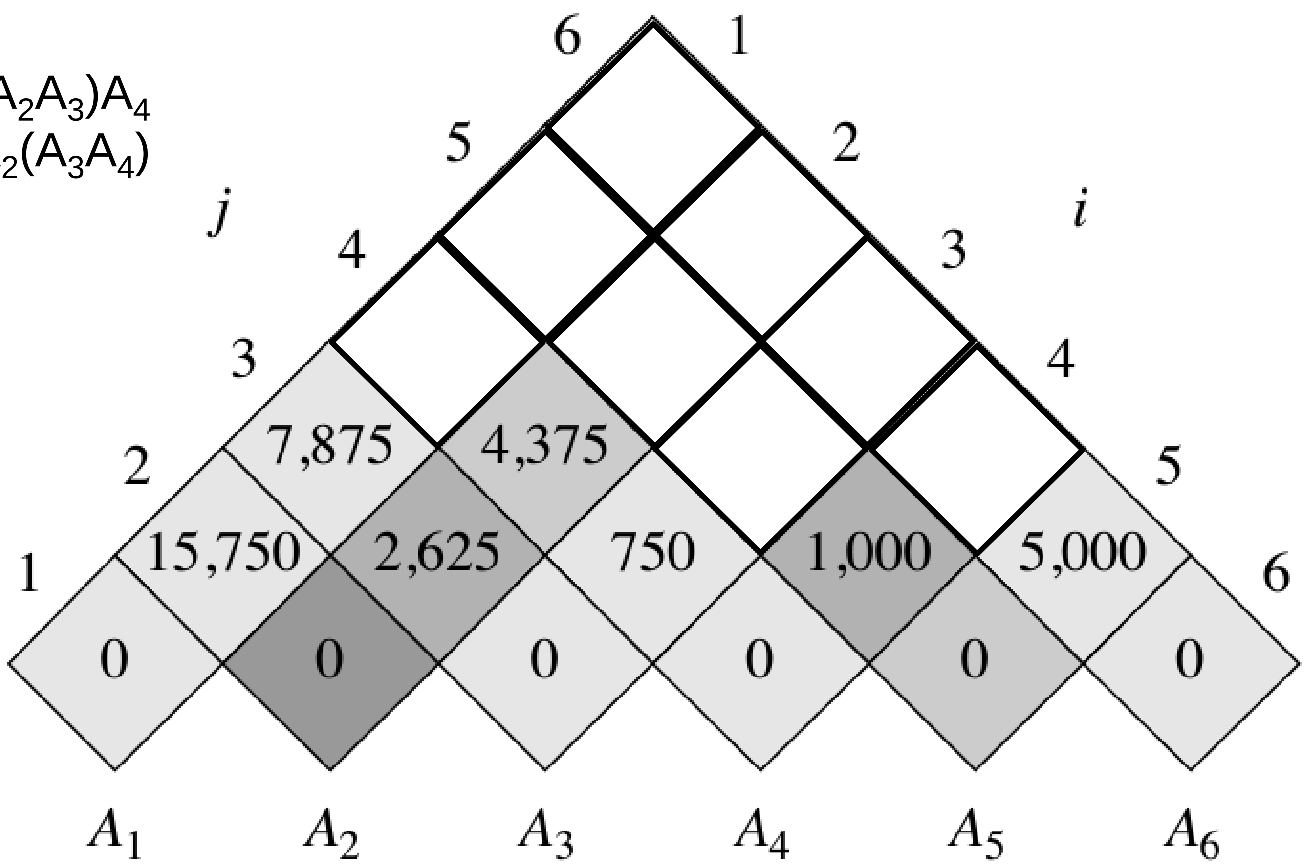
Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n

$A_2A_3A_4 = (A_2A_3)A_4$
Or $A_2(A_3A_4)$



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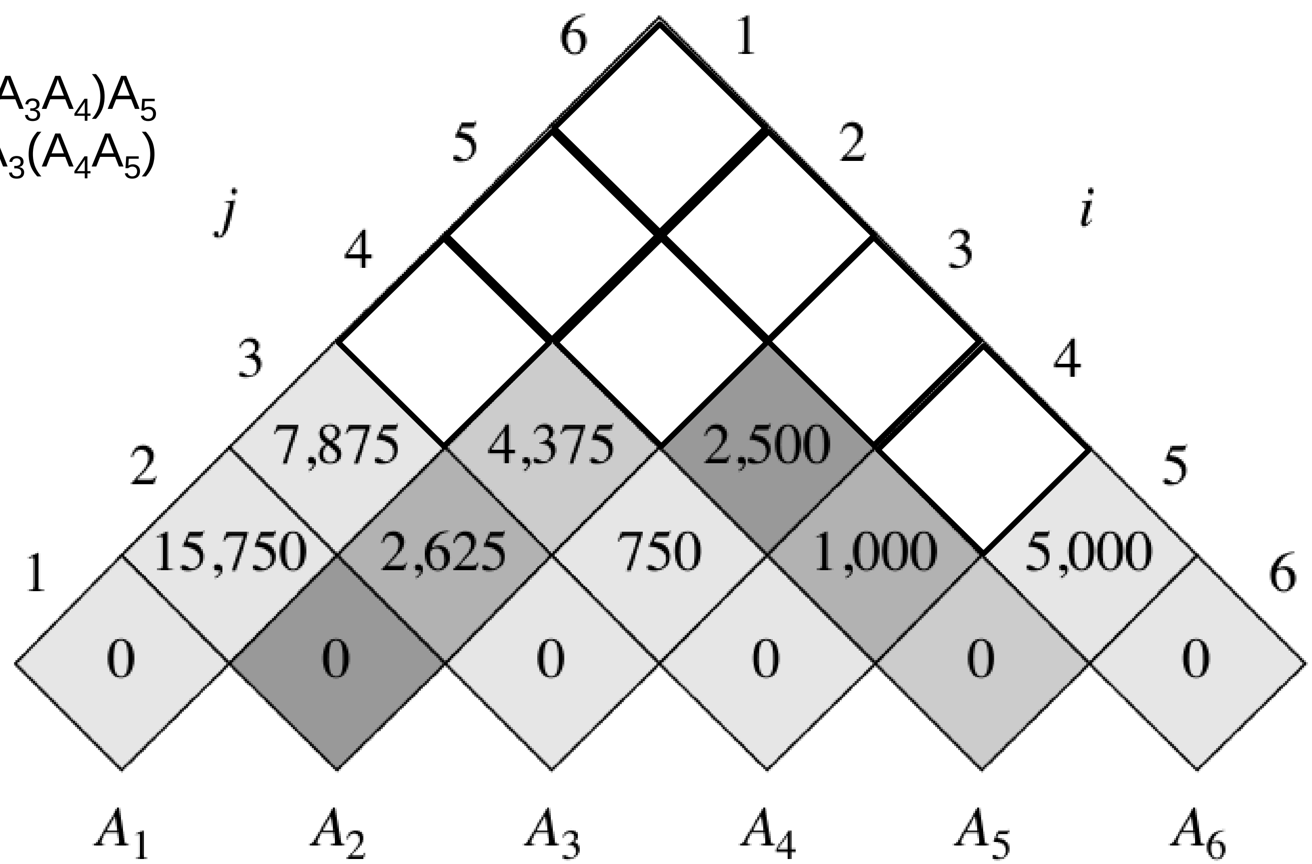
Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n

$A_3A_4A_5 = (A_3A_4)A_5$
Or $A_3(A_4A_5)$



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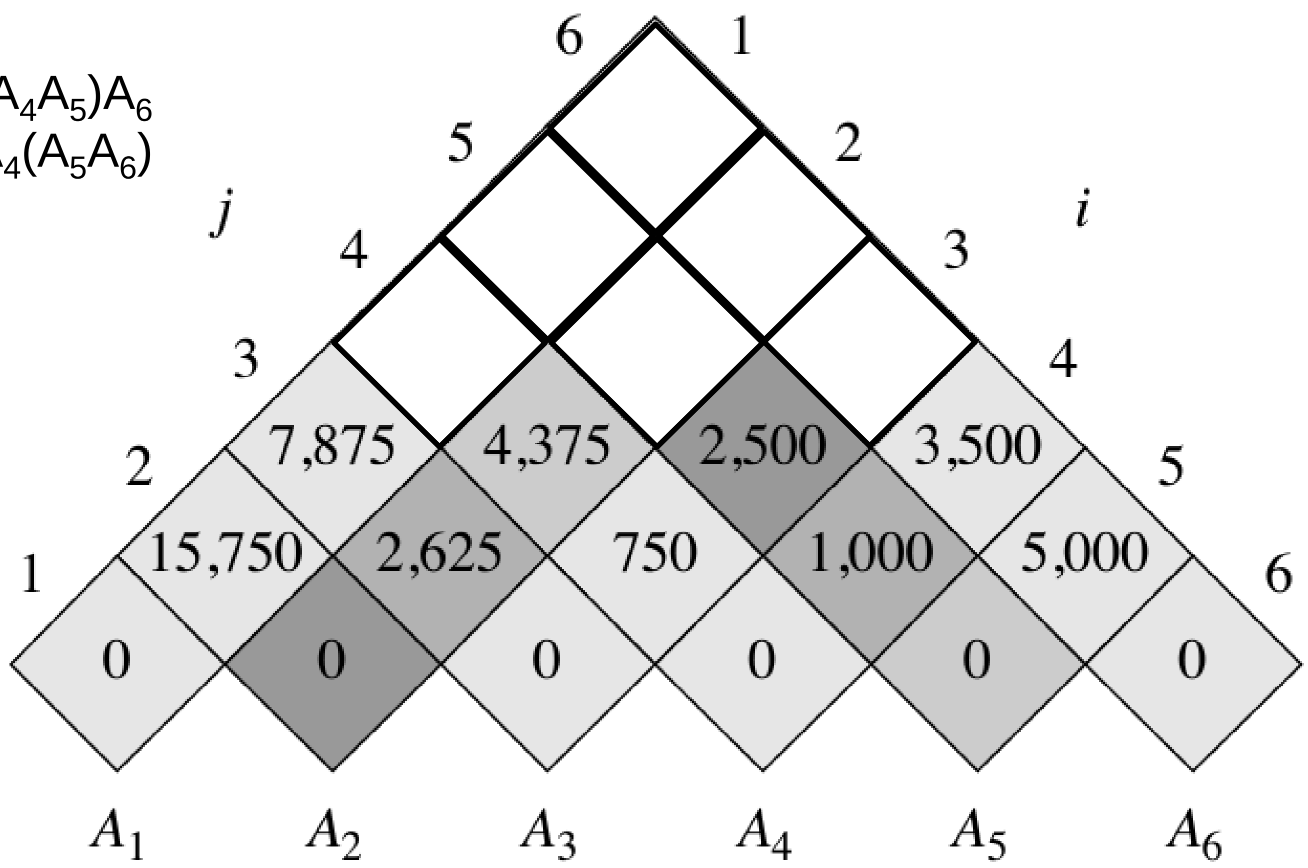
Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n

$A_4A_5A_6 = (A_4A_5)A_6$
Or $A_4(A_5A_6)$



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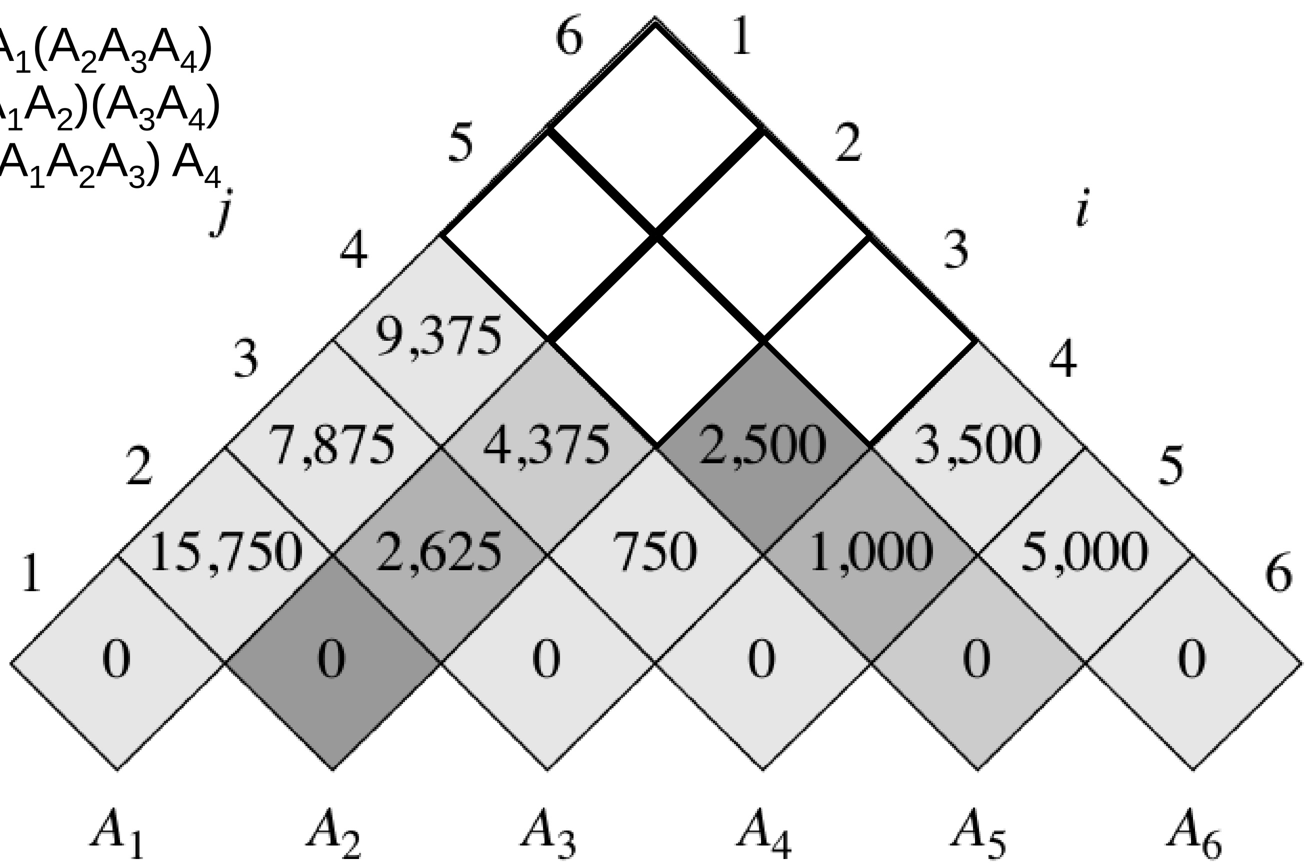
Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n

$A_1A_2A_3A_4 = A_1(A_2A_3A_4)$
Or $(A_1A_2)(A_3A_4)$
Or $(A_1A_2A_3)\ A_4$



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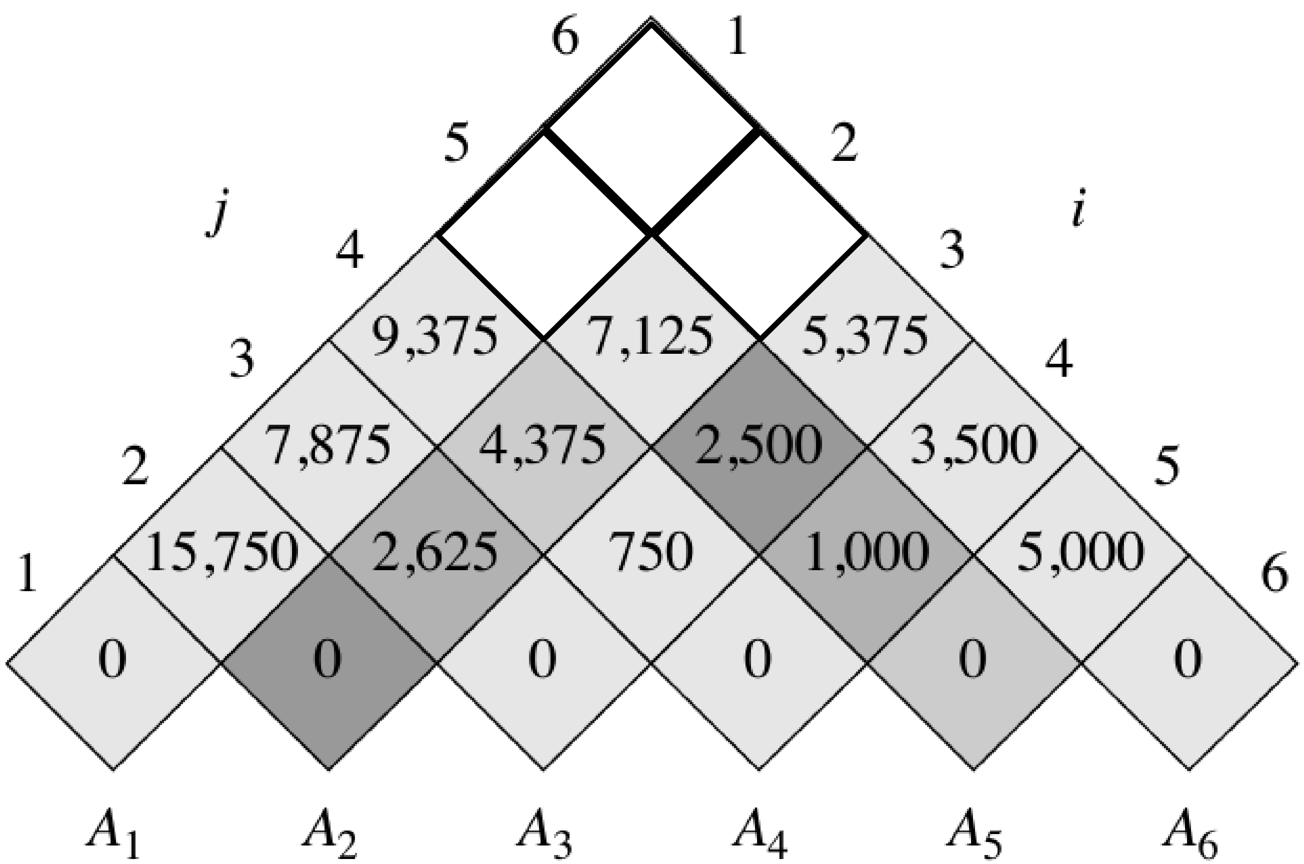

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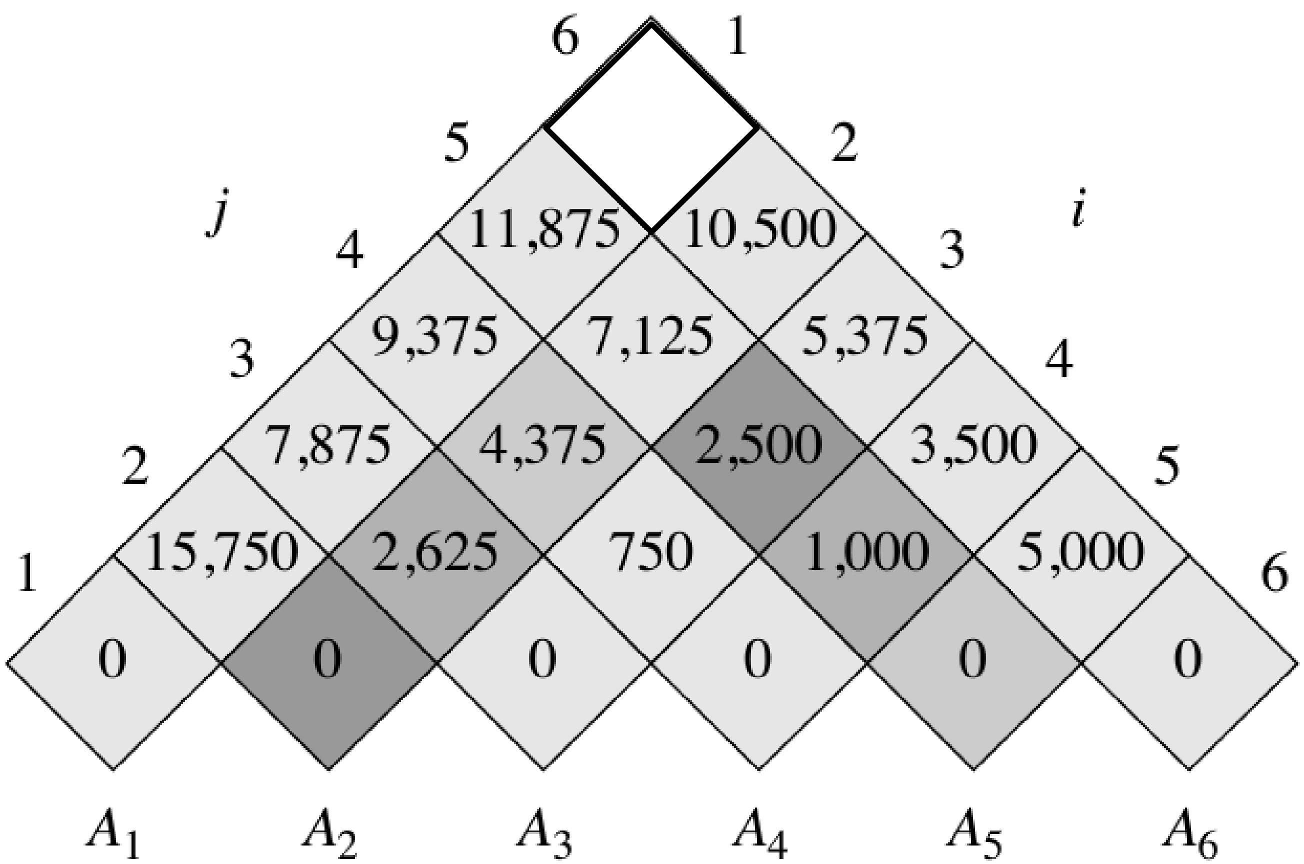

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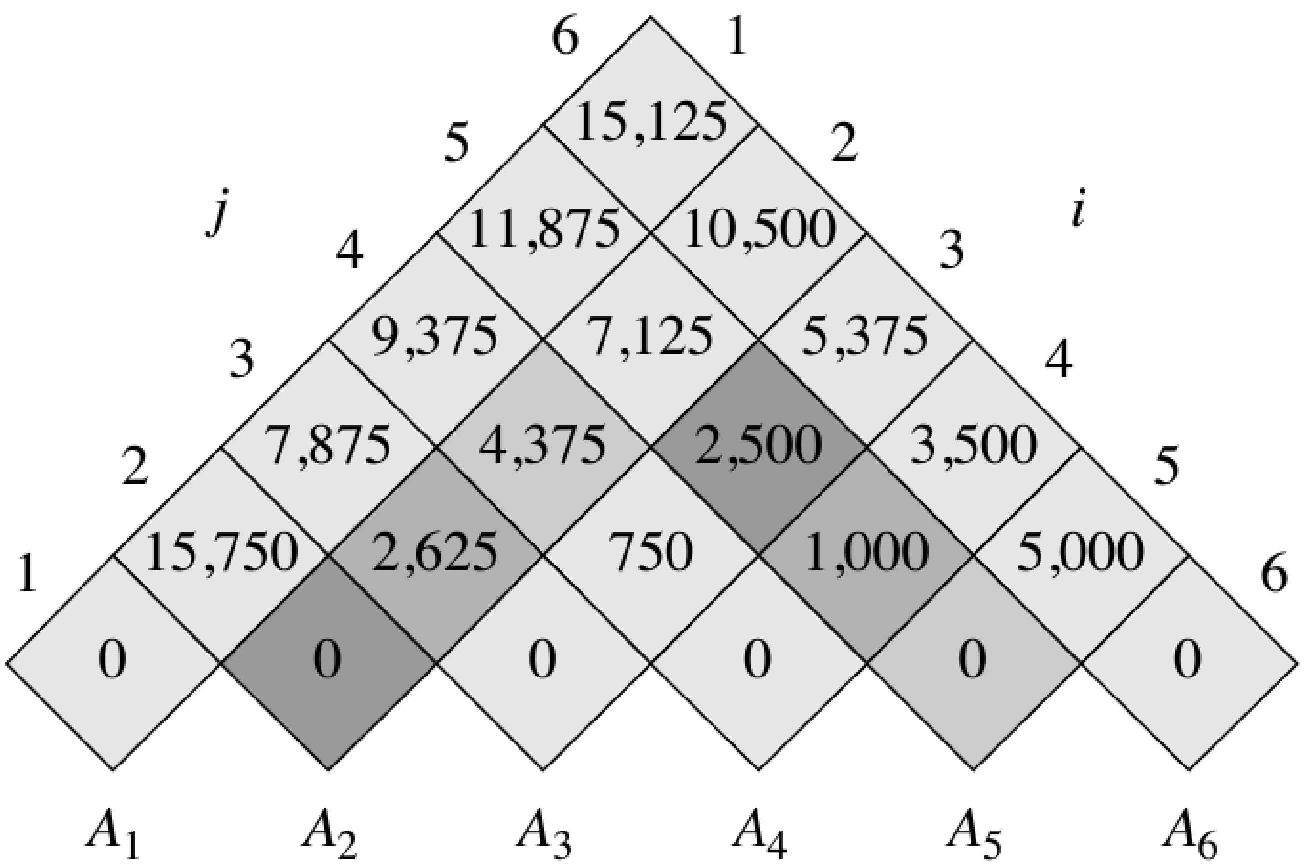

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Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2
- Solve all chains of length 2, then 3, then 4, …n

	1	2	3	4	5	6
1	0	15750	7875	9375	11875	15125
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

A1: 30x35
A2: 35x15
A3: 15x5
A4: 5x10
A5: 10x20
A6: 20x25

Matrix Chain Multiplication

- Generally: $A_i \dots A_k \dots A_j$ of dimensions $p_i \dots p_k \dots p_j$
- $(A_i \dots A_k)(A_{k+1} \dots A_j)$, where k=i,i+1,…j-1
- Then, solve each sub-chains recursively

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- $(A_i \dots A_k)(A_{k+1} \dots A_j)$, where k=i,i+1,…j-1
- Then, solve each sub-chains recursively

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1} p_k p_j\} & \text{if } i < j \end{cases}$$

Matrix Chain Multiplication

MATRIX-CHAIN-ORDER(p)

```
n = p.length − 1
let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
for i = 1 to n
    m[i, i] = 0
for l = 2 to n                    // l is the chain length
    for i = 1 to n − l + 1
        j = i + l − 1
        m[i, j] = ∞
        for k = i to j − 1
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q
                s[i, j] = k
return m and s
```

Longest Common Subsequence

$$S_1 = \text{ACCGGTCGAGTGCGCGGAAGCCGGCCGAA}$$
$$S_2 = \text{GTCGTTCGGAATGCCGTTGCTCTGTAAA}$$

- A string subsequence is an ordered set of characters (not necessarily consecutive)
- A common subsequence of two strings is a subsequence that exist in both strings.
- The longest common subsequence is the common subsequence of the maximum length.

Longest Common Subsequence

- Given two strings:

$$X = \langle x_1, x_2, \ldots, x_m \rangle$$
$$Y = \langle y_1, y_2, \ldots, y_n \rangle$$

- Find the longest common subsequence of X and Y
  LCS(X,Y)

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  - Brute force?

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  - Brute force? $O(n*2^m)$ or $O(m*2^n)$ [enumerate all subsequences of X and check in Y, or vice versa]

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  - If $x_m = y_n$, then this character can be included in LCS, and then we get LCS of X without x_m and Y without y_n (prefix X_{m-1} and Y_{n-1})

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- Are smaller problems simpler?
  - If $x_m = y_n$, then this character can be included in LCS, and then we get LCS of X without x_m and Y without y_n (prefix X_{m-1} and Y_{n-1})
  - Let's define string prefixes
    $$X_i = \langle x_1, x_2, \ldots, x_i \rangle, \text{for } i = 0, 1, \ldots, m$$
  - and same for Y_j for j = 0,1,…., n

Longest Common Subsequence

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

- Prove by contradiction

Longest Common Subsequence

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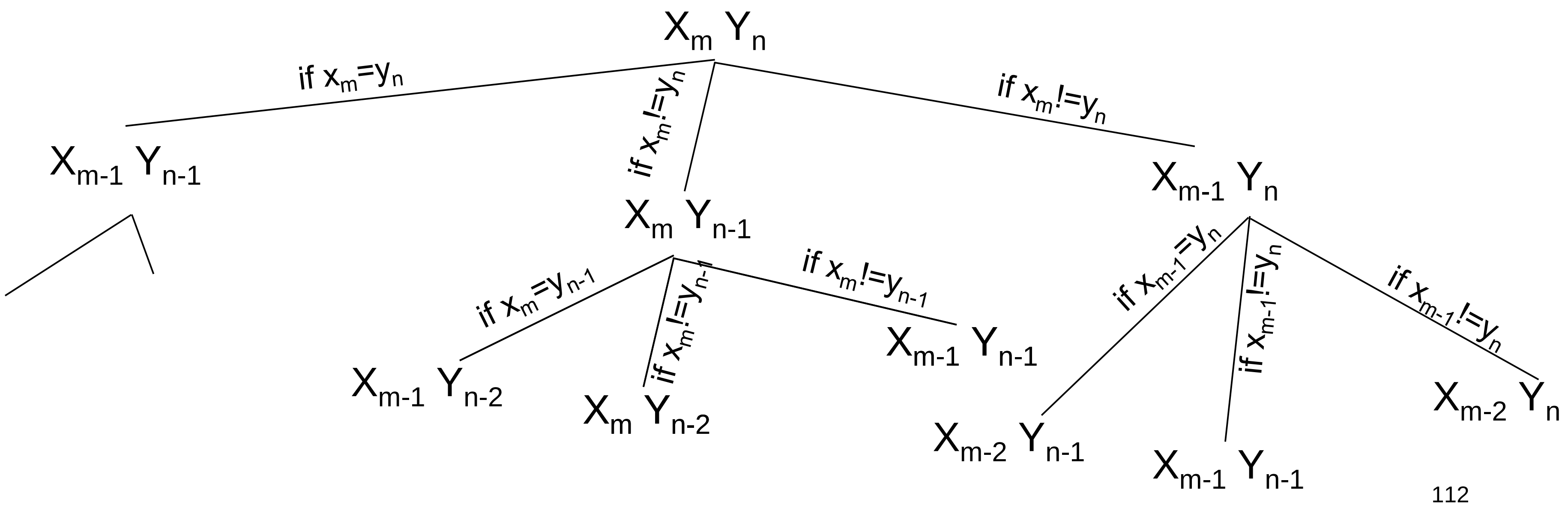

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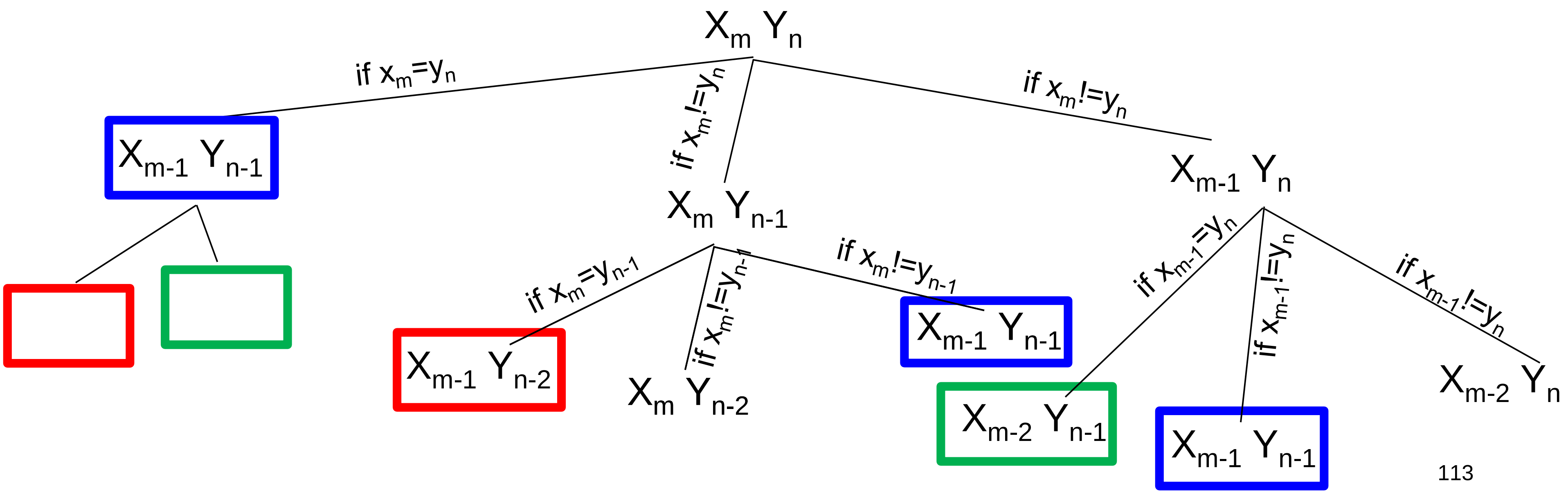

Longest Common Subsequence

- Let c[i,j] is LCS length of X_i and Y_j

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\ , \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j\ , \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j\ . \end{cases}$$

Longest Common Subsequence

- Example: X="CS141" Y="CS111"

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\,, \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j\,, \\ \max(c[i,j-1], c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j\,. \end{cases}$$

	""	C	S	1	1	1
""						
C						
S						
1						
4						
1						

Longest Common Subsequence

- Example: X="CS141" Y="CS111"

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\,, \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j\,, \\ \max(c[i,j-1], c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j\,. \end{cases}$$

	""	C	S	1	1	1
""	0	0	0	0	0	0
C	0					
S	0					
1	0					
4	0					
1	0					

Longest Common Subsequence

- Example: X="CS141" Y="CS111"

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \,, \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j \,, \\ \max(c[i,j-1], c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j \,. \end{cases}$$

	""	C	S	1	1	1
""	0	0	0	0	0	0
C	0	1	1	1	1	1
S	0	1	2	2	2	2
1	0	1	2	3	3	3
4	0	1	2	3	3	3
1	0	1	2	3	4	4

Longest Common Subsequence

```
LCS-LENGTH(X, Y)
m [redacted]ength
n [redacted]ength
let b[1..m, 1..n] and c[0..m, 0..n] be new tables
for i = 1 to m
    c[i, 0] = 0
for j = 0 to n
    c[0, j] = 0
for i = 1 to m
    for j = 1 to n
        if x_i == y_j
            c[i, j] = c[i − 1, j − 1] + 1
            b[i, j] = "↖"
        elseif c[i − 1, j] ≥ c[i, j − 1]
            c[i, j] = c[i − 1, j]
            b[i, j] = "↑"
        else c[i, j] = c[i, j − 1]
            b[i, j] = "←"
return c and b
```

Book Readings

- Ch. 15: 15.1-15.4