CS141: Intermediate Data Structures and Algorithms

Analysis of Algorithms

Amr Magdy

Analyzing Algorithms

1. Algorithm Correctness
    a. Termination
    b. Produces the correct output for all possible input.

2. Algorithm Performance
    a. Either runtime analysis,
    b. or storage (memory) space analysis
    c. or both

Algorithm Correctness

- Sorting problem
  - Input: an array A of n numbers
  - Output: the same array in ascending sorted order (smallest number in A[1] and largest in A[n])

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- Insertion Sort

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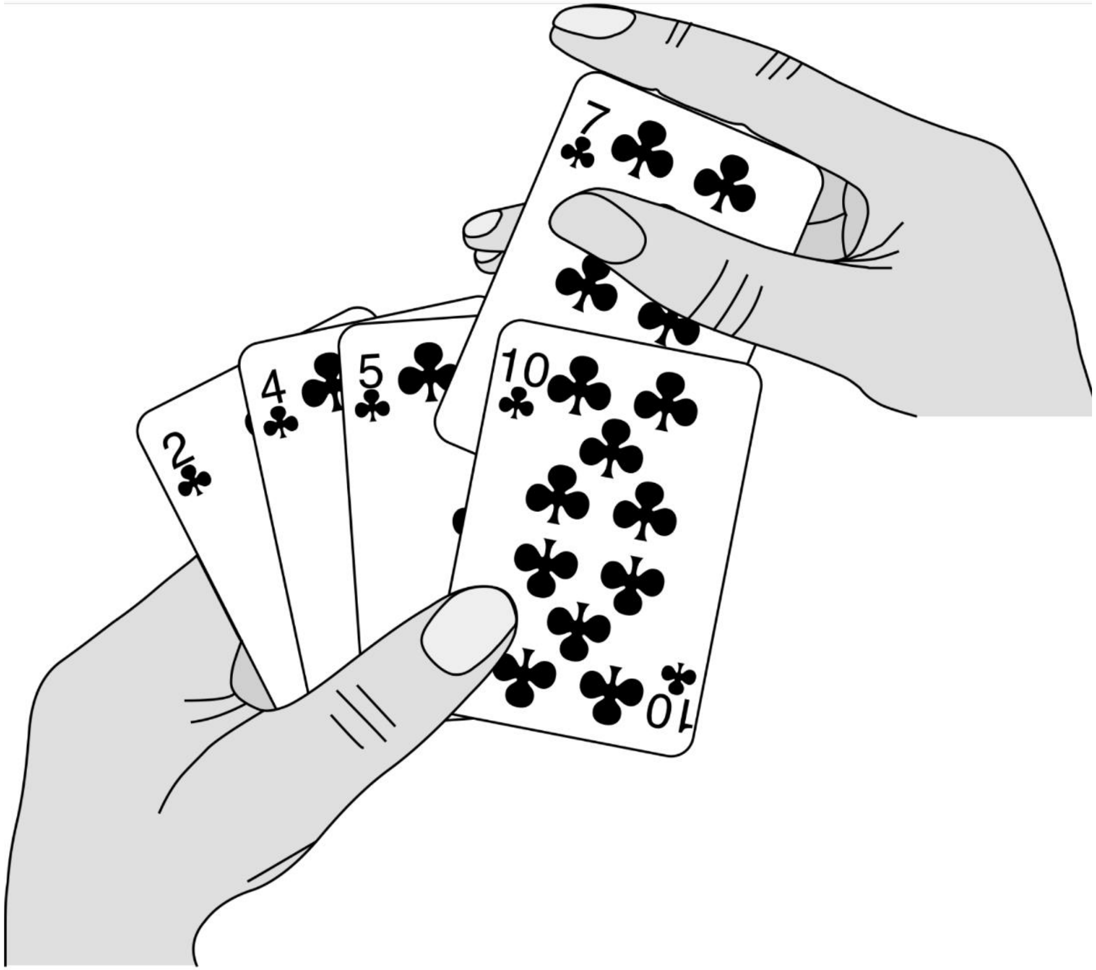
INSERTION-SORT$(A, n)$
  for $j = 2$ to $n$
      $key = A[j]$
      // Insert $A[j]$ into the sorted sequence $A[1..j-1]$.
      $i = j - 1$
      while $i > 0$ and $A[i] > key$
          $A[i+1] = A[i]$
          $i = i - 1$
      $A[i+1] = key$
```

Algorithm Correctness

- How does insertion sort work?



Algorithm Correctness

5	2	4	6	1	3

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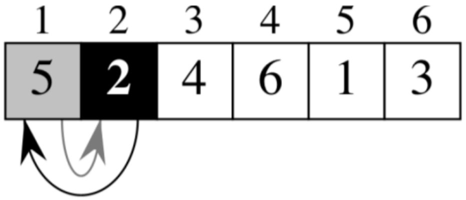

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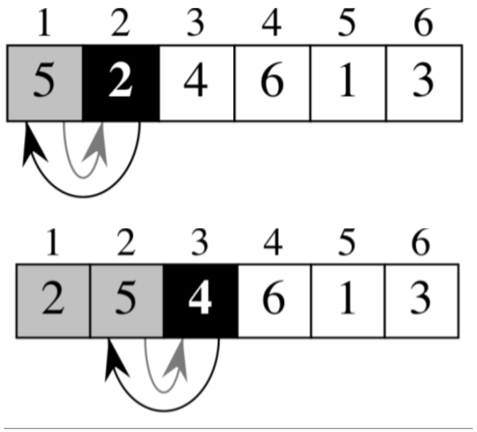

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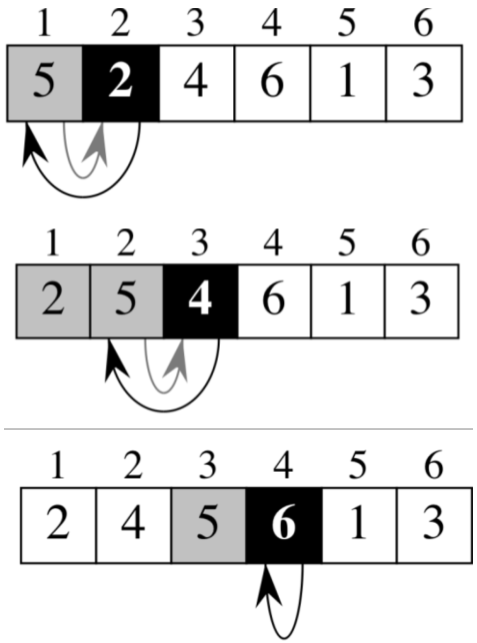

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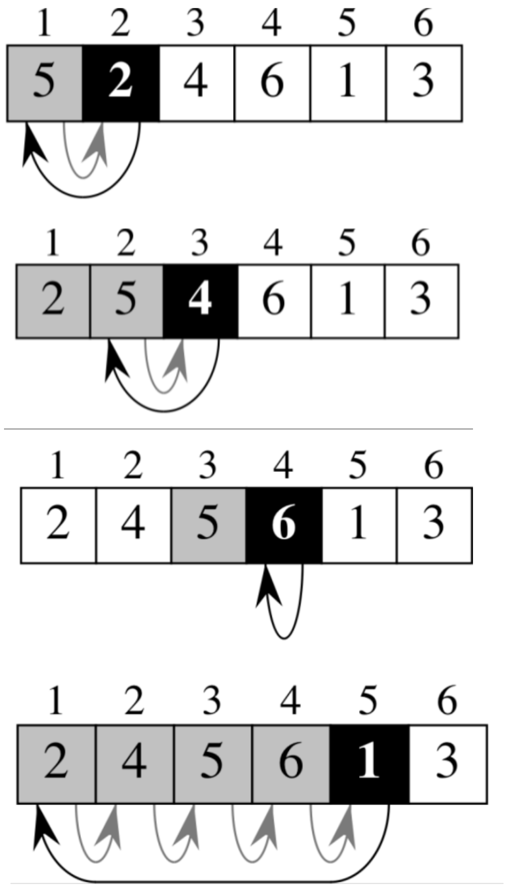

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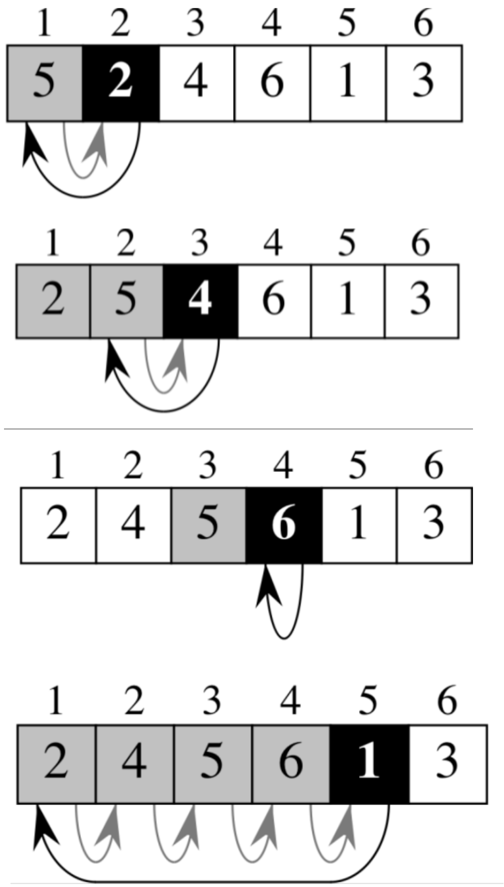

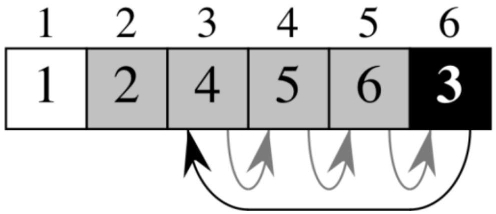

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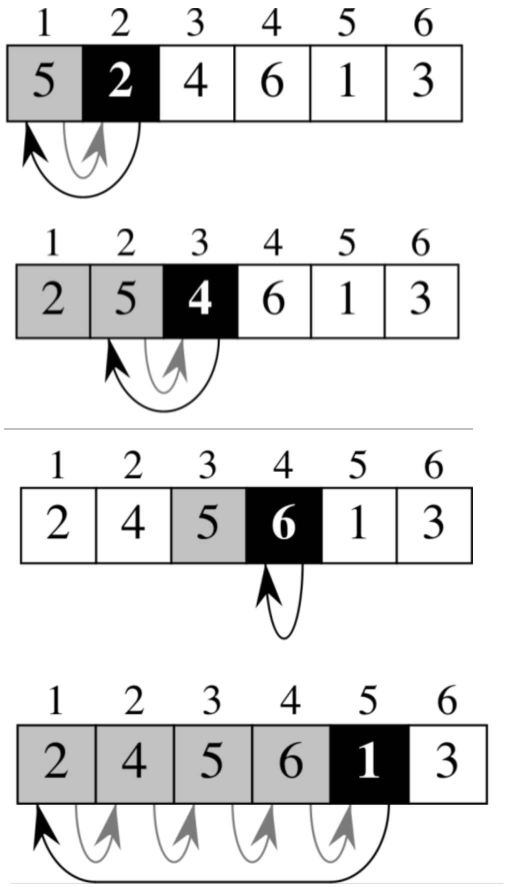

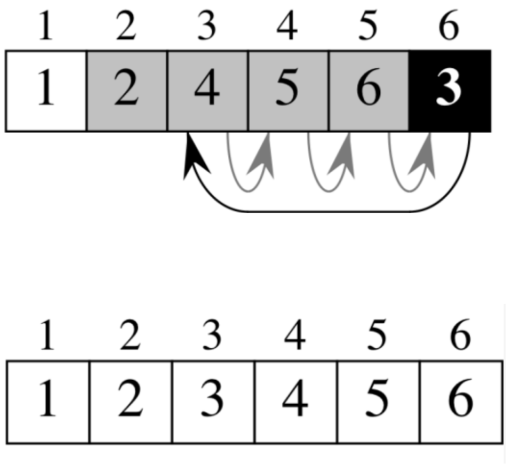

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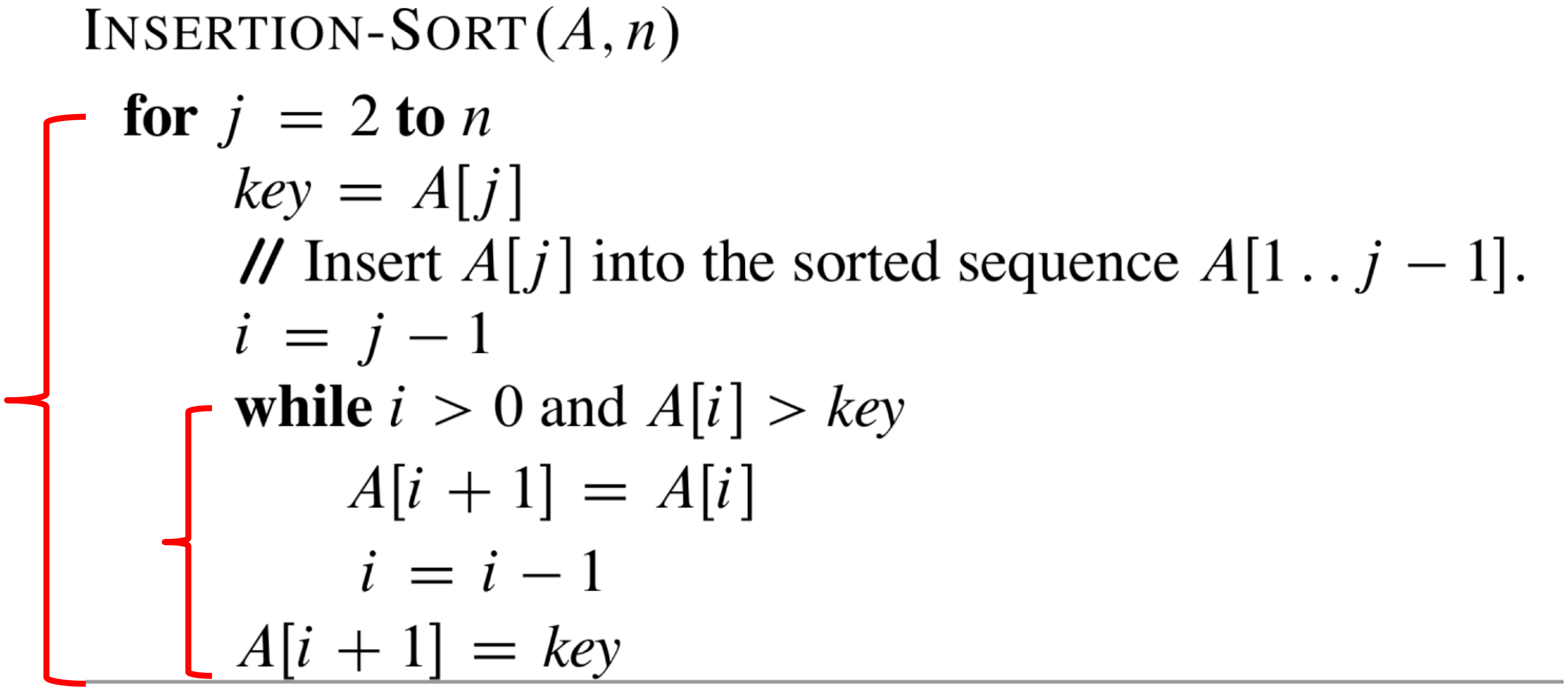

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Algorithm Correctness

- Is insertion sort a correct algorithm?
  - Does it halt? Yes
  - Does it produce correct output for all possible input?
    - Will check through loop invariants for insertion sort
    - For other algorithms, we can use any systematic logic/steps to show that, either loop invariants or other methods

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- Loop invariant:
  - It is a property that is true before and after each loop iteration.

Algorithm Correctness

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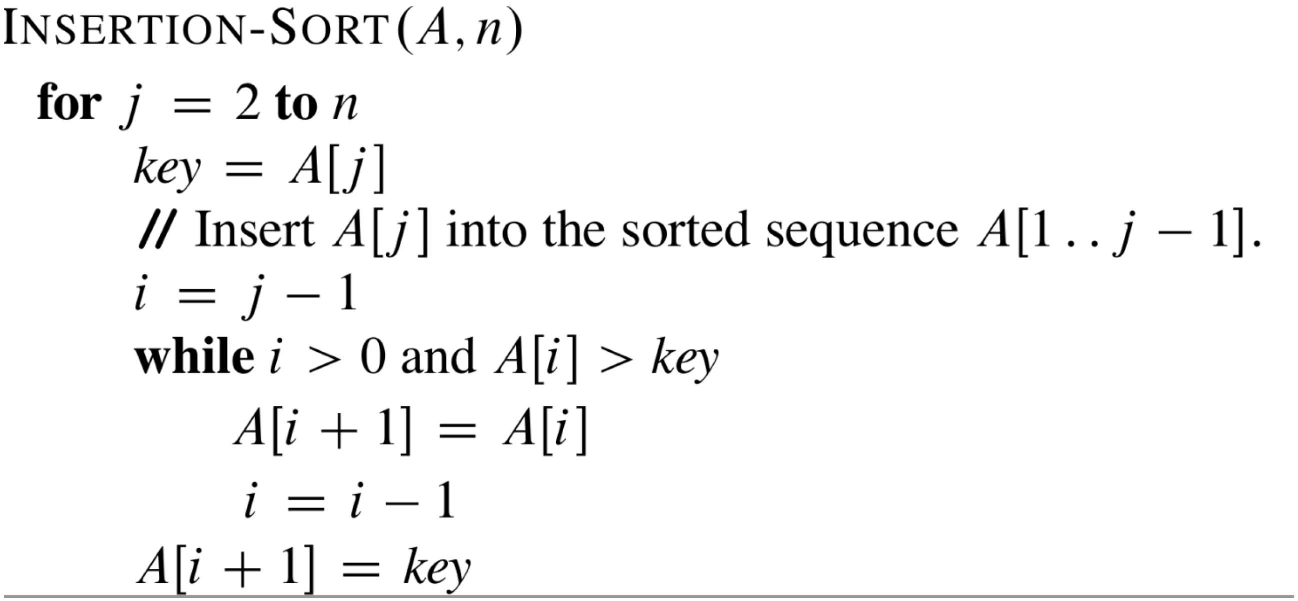
- Loop invariant:
  - It is a property that is true before and after each loop iteration.
- Insertion sort loop invariant (ISLI):
  - The first (j-1) array elements A[1..j-1] are:
    (a) the original (j-1) elements, and (b) sorted.



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- Loop invariant (LI) correctness

1. Initialization:
   LI is true prior to the 1st iteration.
2. Maintenance:
   If LI true before the iteration, it remains true before the next iteration
3. Termination:
   After the loop terminates, the output is correct.

Participation Exercise

Name: Student ID:

Insertion sort loop invariant (ISLI): The first (j-1) array elements A[1..j-1] are:
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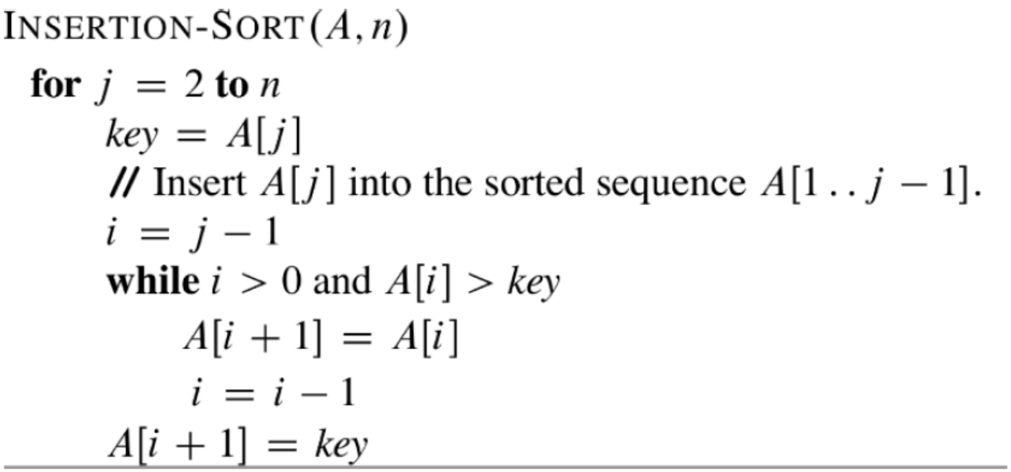

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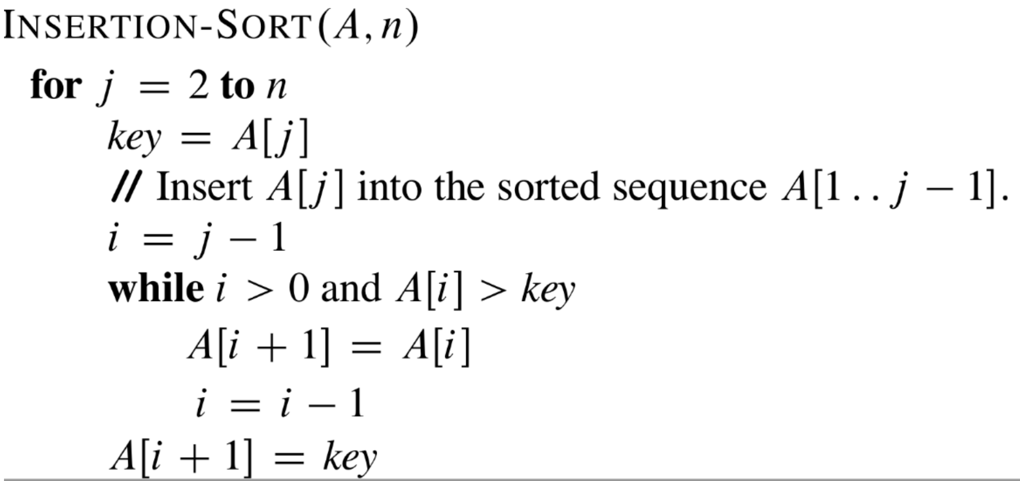
Loop invariant (LI) correctness:

1. Initialization: Is LI true prior to the 1st iteration?
2. Maintenance: If LI true before iteration j, does it remain true after iteration j and before iteration (j+1)?
3. Termination: After the loop terminates, is the output correct?

Answer the three questions for ISLI.

Algorithm Correctness

- ISLI: The first (j-1) array elements A[1..j-1] are:
  (a) the original (j-1) elements, and (b) sorted.
  1. Initialization:
     Prior to the 1st iteration, j=2, the first (2-1)=1 elements is sorted.
  2. Maintenance:
     The (j-1)th iteration inserts the jth element in a sorted order, so after the iteration, the first (j-1) elements remains the same and sorted.
  3. Termination:
     The loop terminates after (n-1) iterations, j=n+1, so the first n elements are sorted, then the output is correct.



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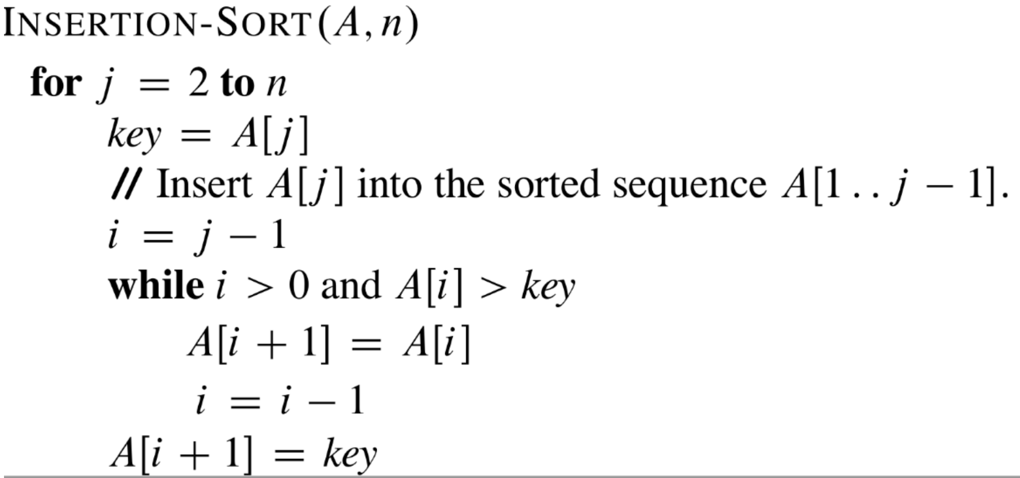

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Correct

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✓

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Algorithms Performance Analysis

- Which criteria should be taken into account?

- Running time
- Memory footprint
- Disk IO
- Network bandwidth
- Power consumption
- Lines of codes
- …

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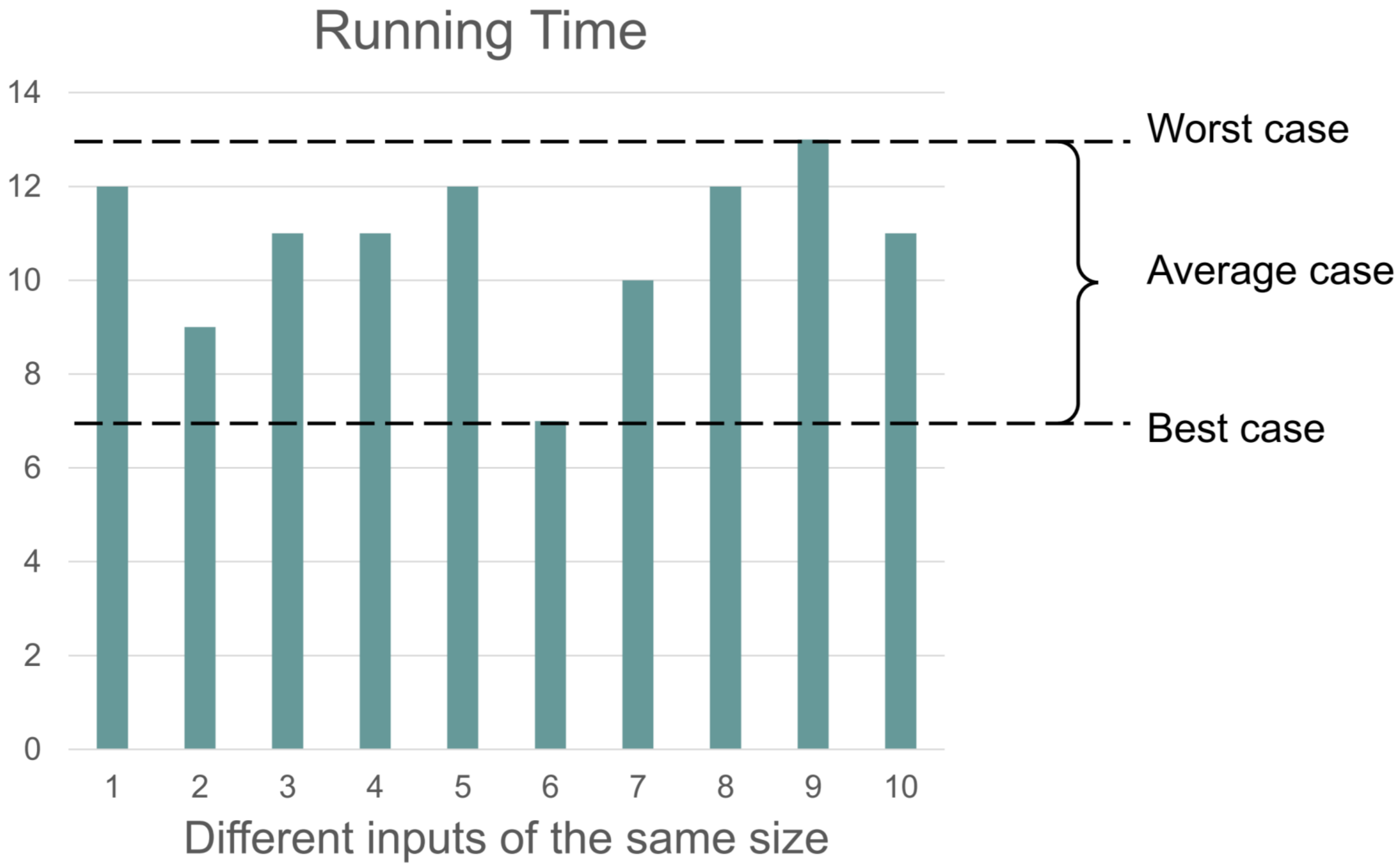
Average Case vs. Worst Case



Insertion Sort Best Case

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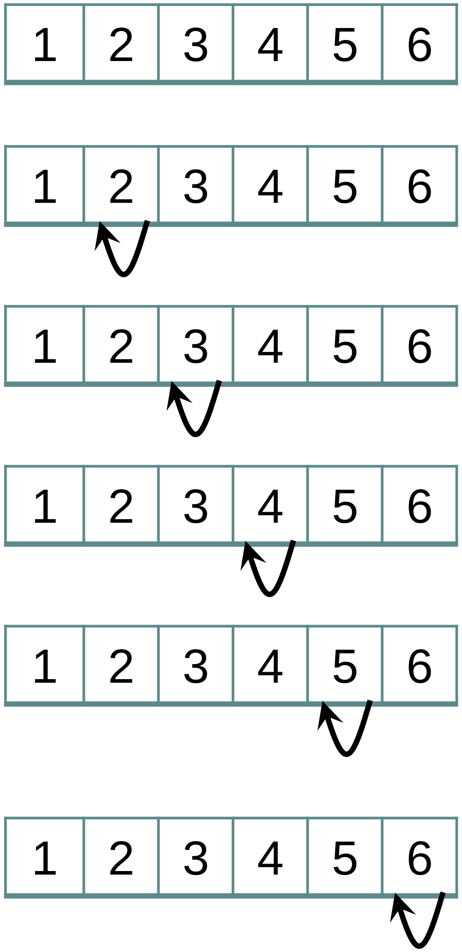
- Input array is sorted



1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
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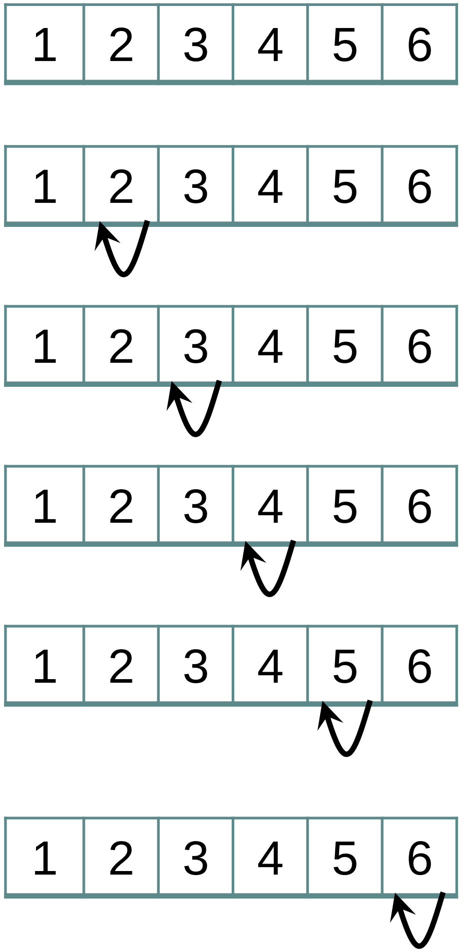

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INSERTION-SORT$(A, n)$
for $j = 2$ to $n$ ..........................................c1
    $key = A[j]$ ..........................................c2
    // Insert $A[j]$ into the sorted sequence $A[1..j-1]$..........0
    $i = j - 1$ ..........................................c3          (n-1)
    while $i > 0$ and $A[i] > key$ ..........................c4
        $A[i+1] = A[i]$
                          do not execute .......... 0        1
        $i = i - 1$
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```

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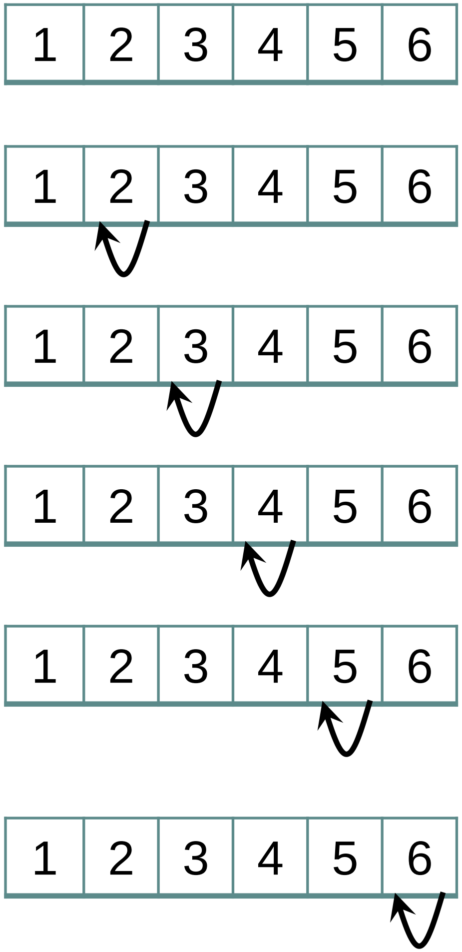

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        $A[i+1] = A[i]$  ┐                                      │ 1    │
        $i = i - 1$      ┘ do not execute .......... 0          ┘      │
    $A[i+1] = key$ ..............................................c5   ┘
```

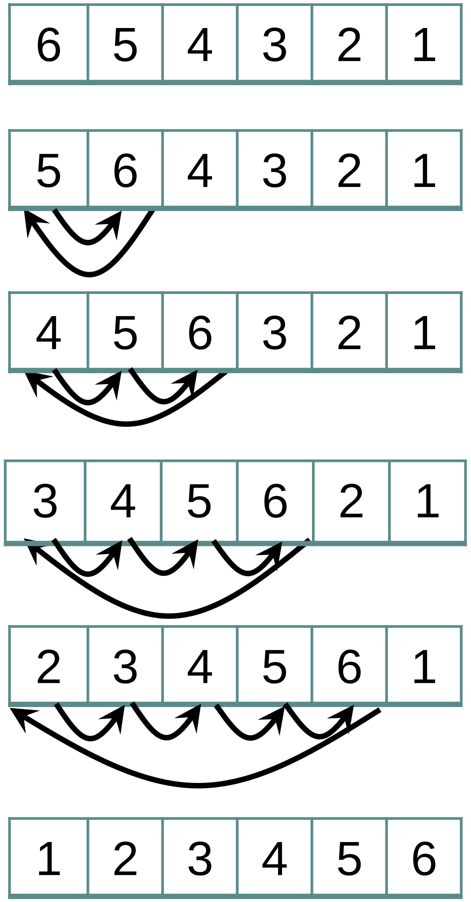
T(n) = (n-1)*(c1+c2+0+c3+1*(c4+0)+c5)

T(n) = cn-c, const c=c1+c2+c3+c4+c5

Insertion Sort Worst Case

Insertion Sort Worst Case

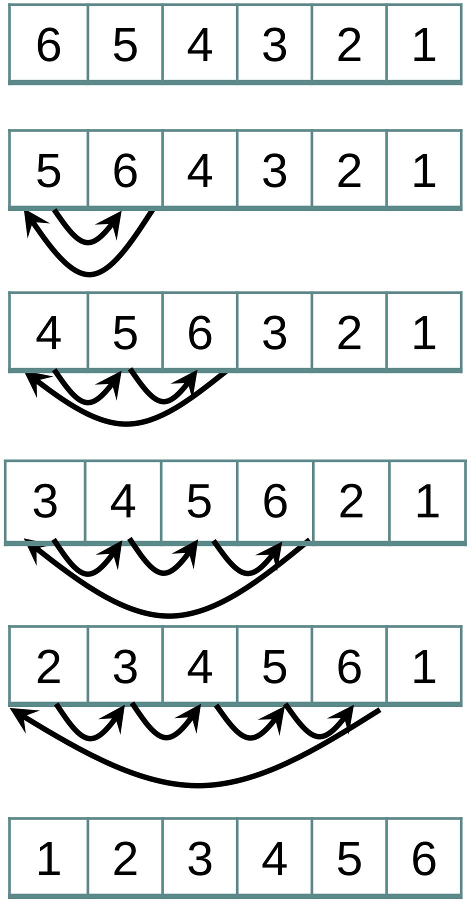
- Input array is reversed



6	5	4	3	2	1
5	6	4	3	2	1
4	5	6	3	2	1
3	4	5	6	2	1
2	3	4	5	6	1
1	2	3	4	5	6

Insertion Sort Worst Case

- Input array is reversed



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    while $i > 0$ and $A[i] > key$ ...........................c4 ┐          │
        $A[i+1] = A[i]$ ...................................c5 ├ i        │
        $i = i - 1$ ...........................................c6 ┘          │
    $A[i+1] = key$ ...............................................c7  ┘
```

Insertion Sort Worst Case

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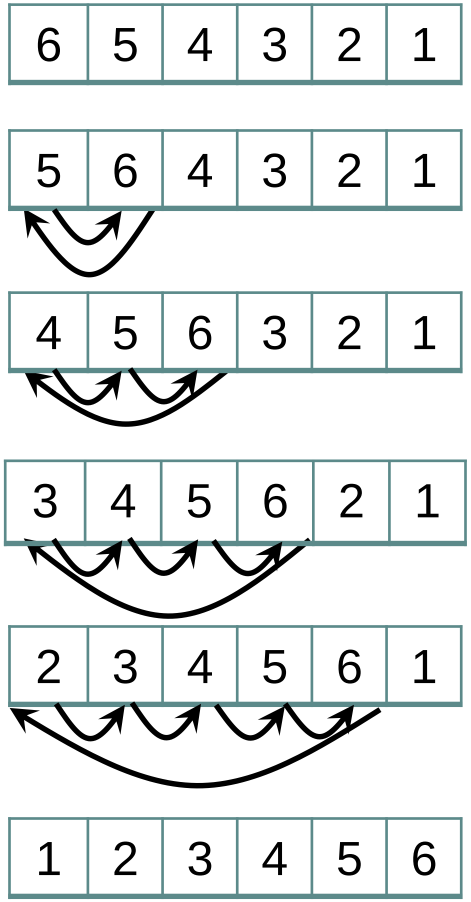

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```

T(n) = (n-1)*(c1+c2+0+c3+i*(c4+c5+c6)+c7)

T(n) = (n-1)*(c1+c2+0+c3+c7) + ∑i*(c4+c5+c6), for all 1 <= i < n

T(n) = (cn-c) + ∑i*d, c & d are constants

∑i*d = 1*d+2*d+3*d+….+(n-1)*d= d *(1+2+3+…(n-1))= d*n(n-1)/2

T(n) = (cn-c) + dn^2/2-dn/2 = d*n^2+c11*n+c12, c's & d are consts

Insertion Sort Average Case

- Average = (Best + Worst)/2
- $T(n) = cn^2+dn+e$,   c, d, e are consts

Which case we consider?

Which case we consider?

- The worst case

Which case we consider?

- The worst case
  - Why?

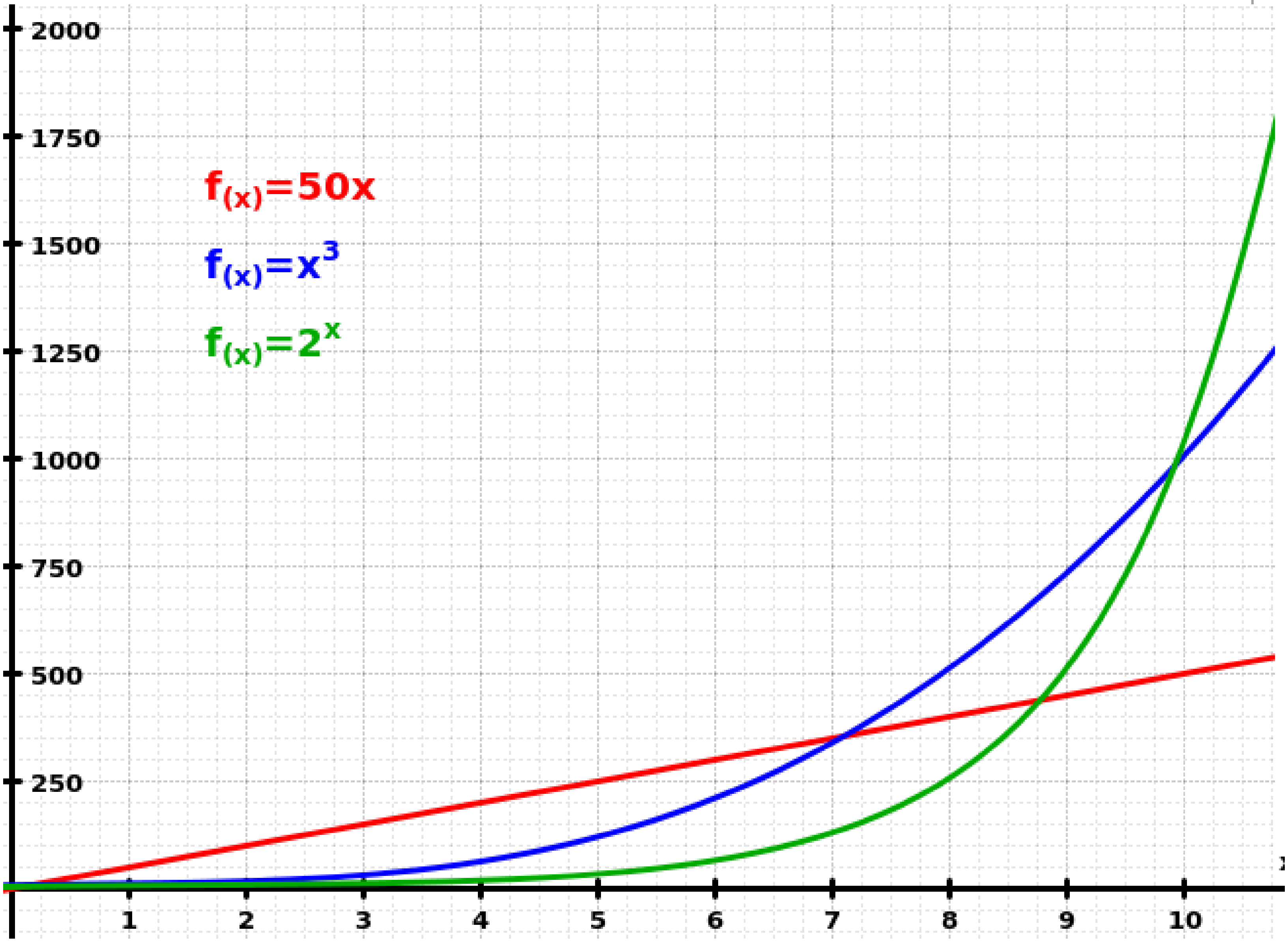
Which case we consider?

- The worst case
  - Why?
    - It gives guarantees on the upper bound performance

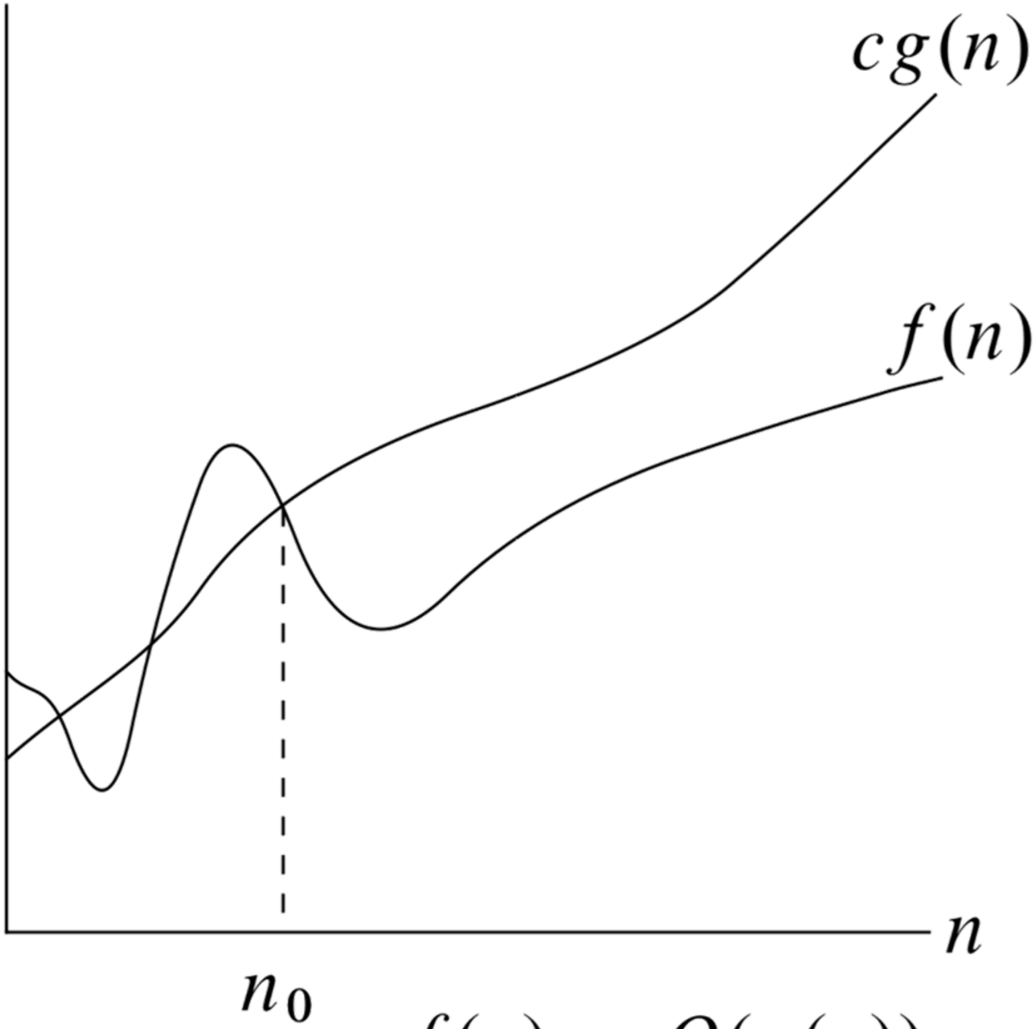
Growth of Functions

- It is hard to compute the actual running time for more complex algorithms
- The cost of the worst-case is a good measure
- The growth of the cost function is what interests us (when input size is large)
- We are more concerned with comparing two cost functions, i.e., two algorithms.

Growth of Functions



O-notation

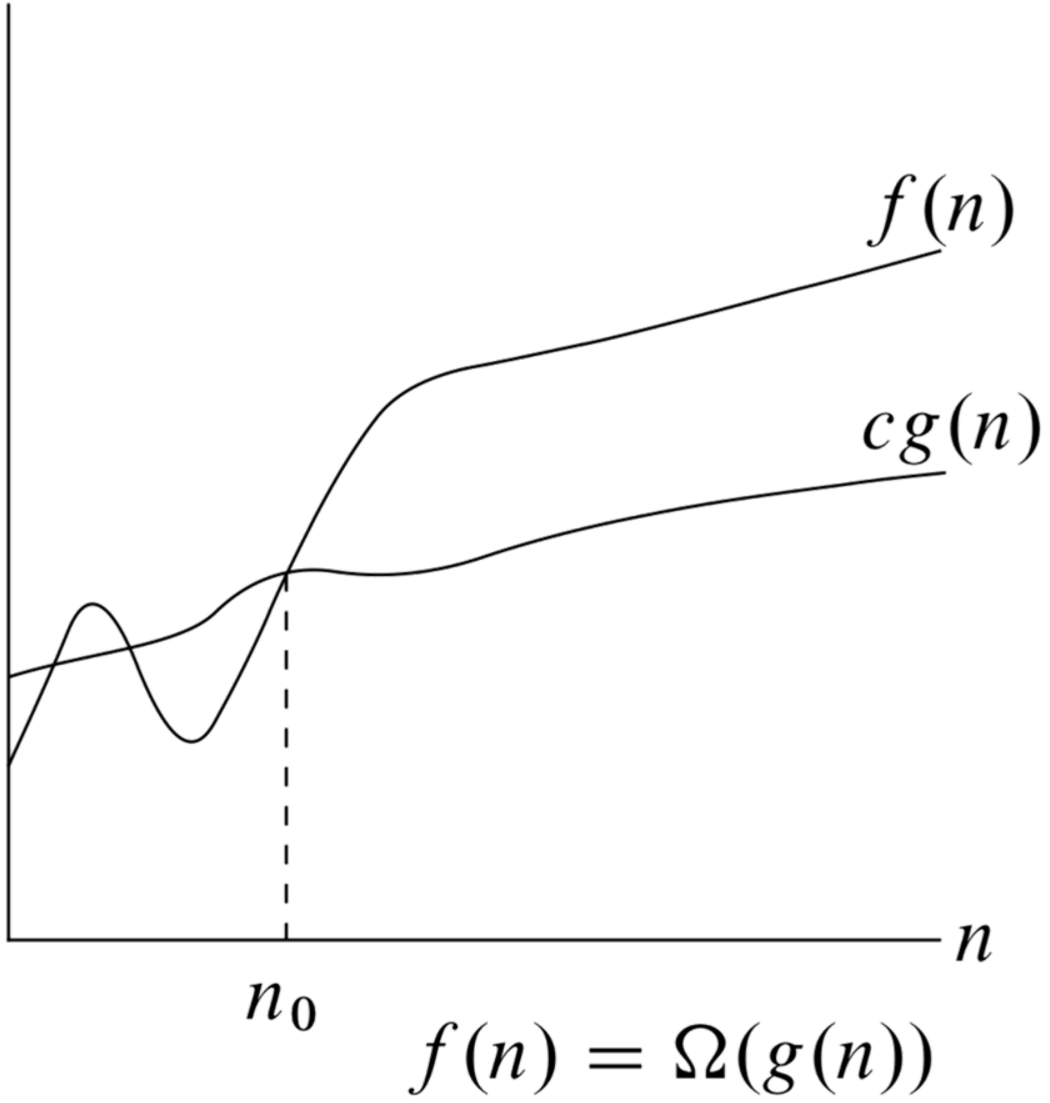


$$f(n) = O(g(n))$$

$$\exists c > 0, n_0 > 0$$
$$0 \le f(n) \le cg(n)$$
$$n \ge n_0$$

g(n) is an asymptotic upper-bound for f(n)

Ω-notation



$f(n) = \Omega(g(n))$

$$\exists c > 0, n_0 > 0$$
$$0 \leq cg(n) \leq f(n)$$
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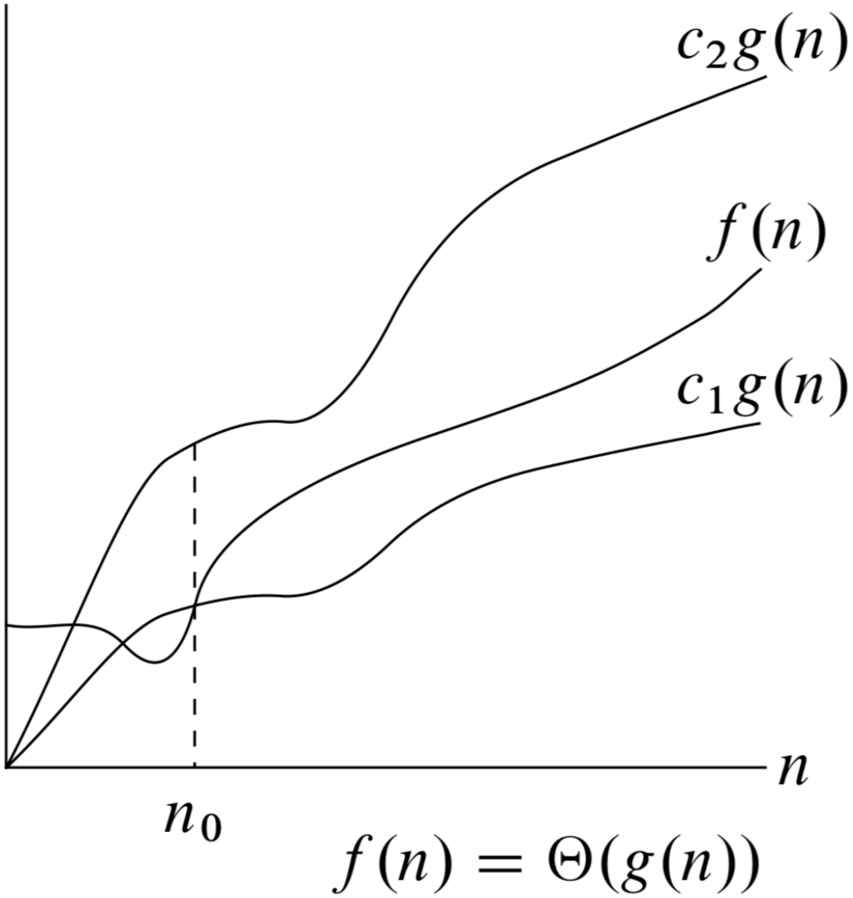
Θ-notation



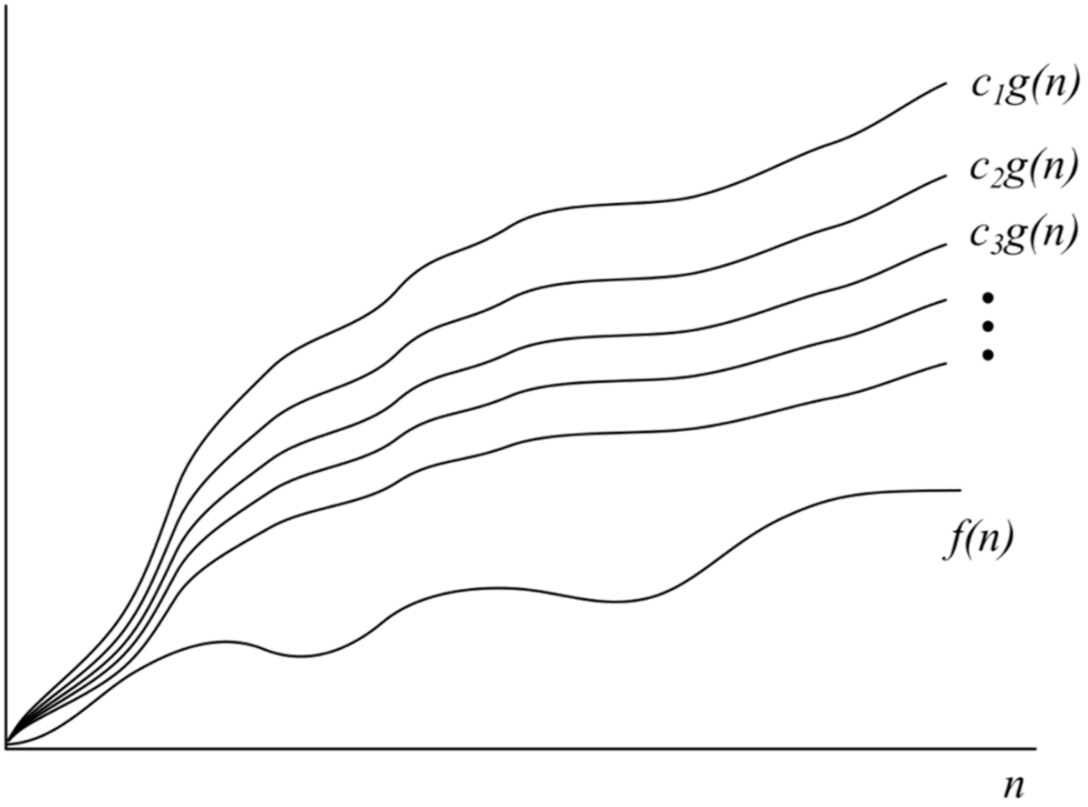
$f(n) = \Theta(g(n))$

$$\exists c_1, c_2 > 0, n_0 > 0$$
$$0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$$
$$n \geq n_0$$

g(n) is an asymptotic tight-bound for f(n)

o-notation

$$\boxed{\forall c > 0}$$
$$\exists n_0 > 0$$
$$0 \leq f(n) \leq cg(n)$$
$$n \geq n_0$$



$$f(n) = o(g(n))$$

g(n) is a non-tight asymptotic upper-bound for f(n)

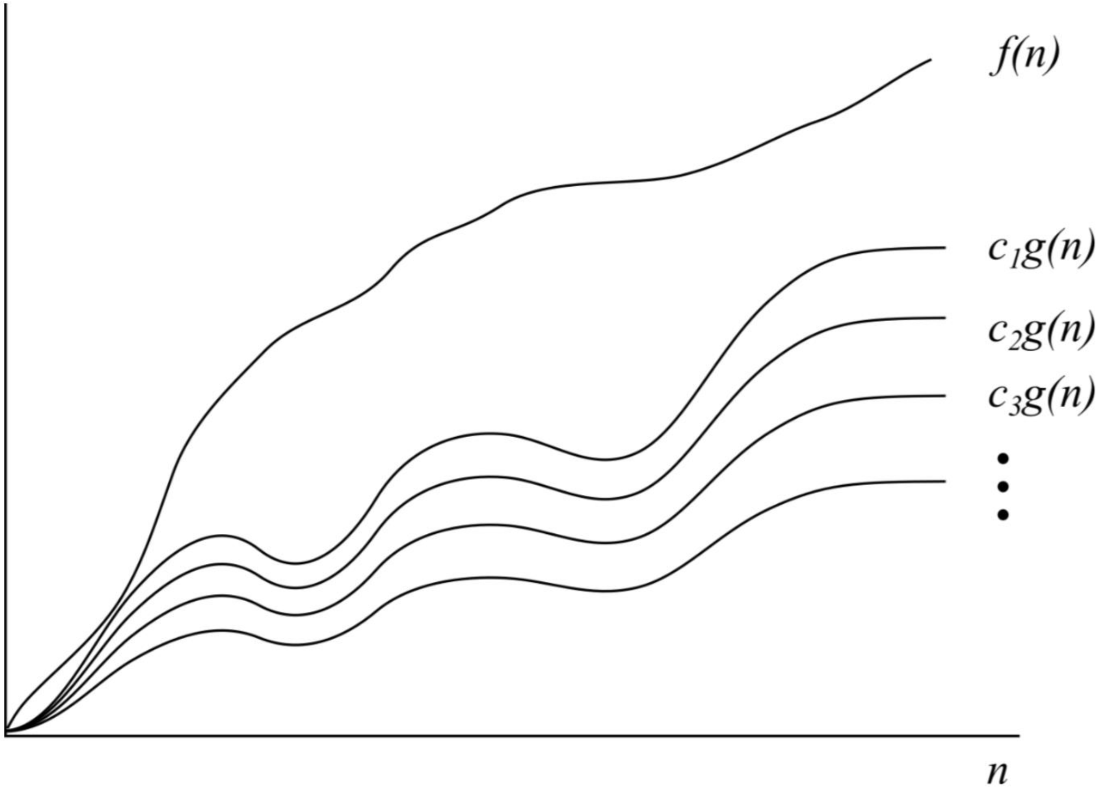
ω-notation

$\forall c > 0$

$\exists n_0 > 0$

$0 \leq cgn(n) \leq f(n)$

$n \geq n_0$



$$f(n) = \omega(g(n))$$

g(n) is a non-tight asymptotic lower-bound for f(n)

Comparing Two Functions

- $lim_{n\rightarrow\infty}\frac{f(n)}{g(n)}$
- 0: f(n) = o(g(n))
- c > 0: f(n) = Θ(g(n))
- ∞: f(n) = ω(g(n))

Analogy to Real Numbers

Functions	Real numbers
$f(n) = O(g(n))$	$a \leq b$
$f(n) = \Omega(g(n))$	$a \geq b$
$f(n) = \Theta(g(n))$	$a = b$
$f(n) = o(g(n))$	$a < b$
$f(n) = \omega(g(n))$	$a > b$

Simple Rules

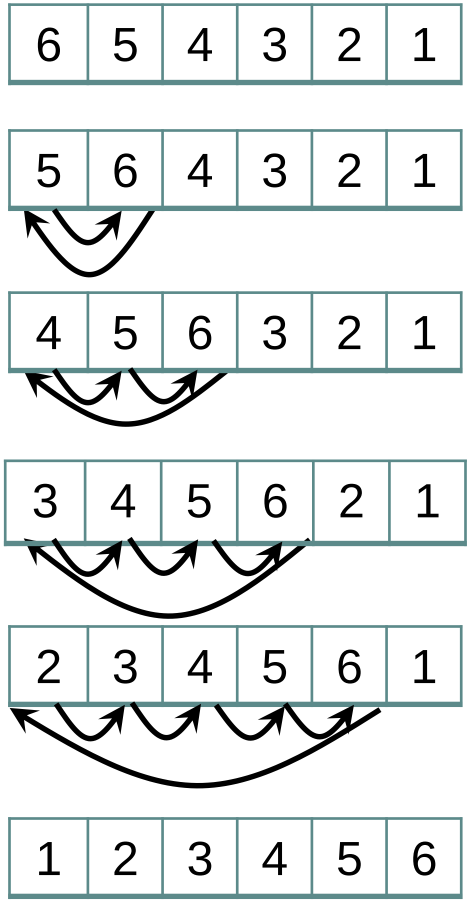
- We can omit constants
- We can omit lower order terms
- Θ(an^2+bn+c) becomes Θ(n^2), a, b, c are constants
- Θ(c1) and Θ(c2) become Θ(1), c’s are constants
- Θ($\log_{k1} n$) and Θ($\log_{k2} n$) become Θ($\log n$), k’s are constants
- Θ($\log(n^k)$) becomes Θ($\log n$), k is constant

Popular Classes of Functions

Constant: $f(n) = \Theta(1)$
Logarithmic: $f(n) = \Theta(\lg(n))$
Sublinear: $f(n) = o(n)$
Linear: $f(n) = \Theta(n)$
Super-linear: $f(n) = \omega(n)$
Quadratic: $f(n) = \Theta(n^2)$
Polynomial: $f(n) = \Theta(n^k)$; k is a constant
Exponential: $f(n) = \Theta(k^n)$; k is a constant

Insertion Sort Worst Case (Revisit)

- Input array is reversed



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max n

(n-1)

T(n) = (n-1)*n = O(n^2)

Comparing two algorithms

- $T1(n) = 2n+1000000$
- $T2(n) = 200n + 1000$
- Which is better? Why?
  - In terms of order of growth?

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Comparing two algorithms

- T1(n) = 2n+1000000
- T2(n) = 200n + 1000
- Which is better? Why?
  - In terms of order of growth? Same
  - In terms of actual runtime?

Comparing two algorithms

- T1(n) = 2n+1000000
- T2(n) = 200n + 1000
- Which is better? Why?
  - In terms of order of growth? Same
  - In terms of actual runtime?
    - For n <= 5045, T2 is faster, otherwise T1 is faster

Comparing two algorithms

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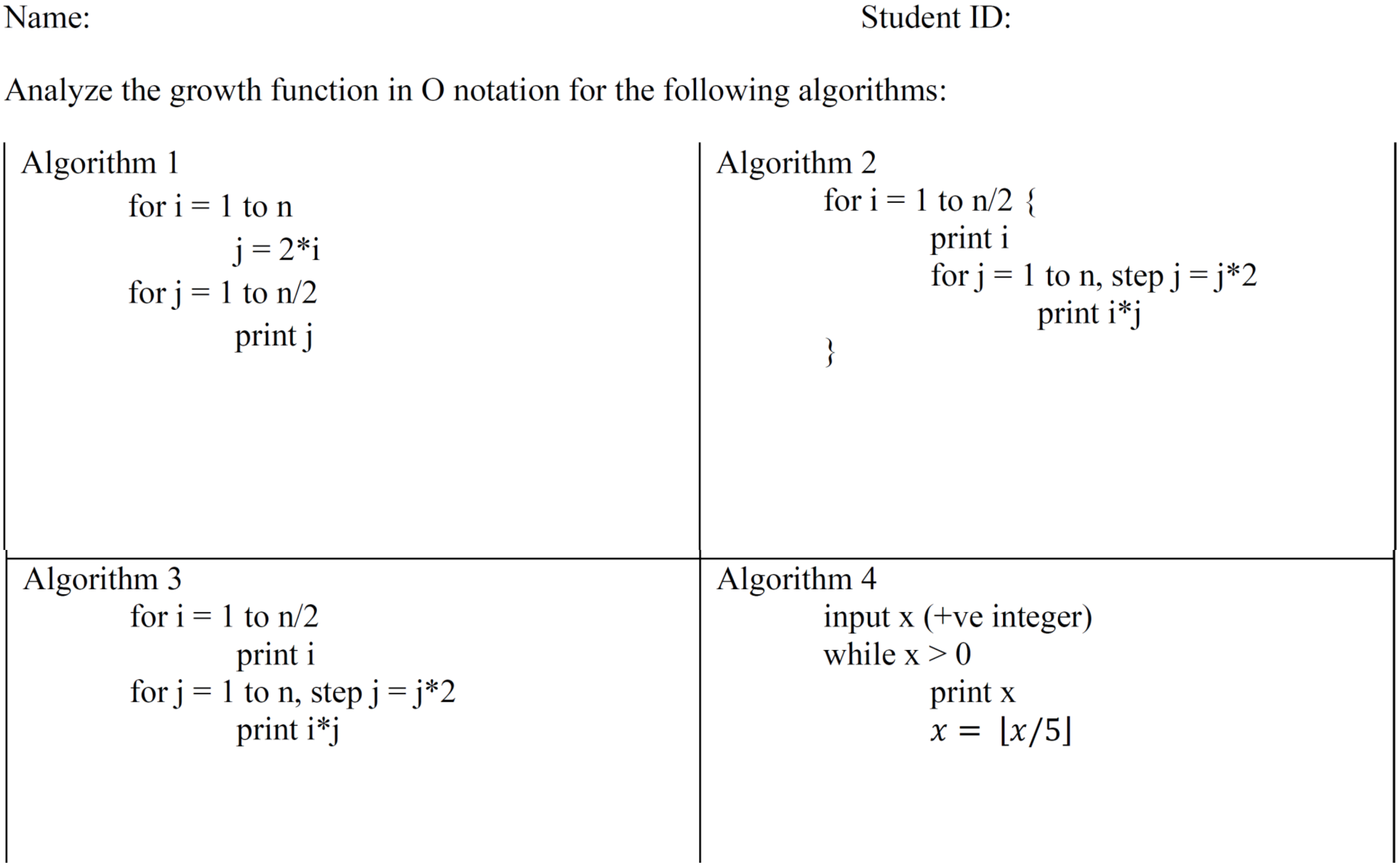
- Which is better? Why?
  - In terms of order of growth? Same
  - In terms of actual runtime?
    - For n <= 5045, T2 is faster, otherwise T1 is faster
- What is the main usage of asymptotic notation analysis?

Participation Exercise



Name: Student ID:

Analyze the growth function in O notation for the following algorithms:

Algorithm 1

```
for i = 1 to n
    j = 2*i
for j = 1 to n/2
    print j
```

Algorithm 2

```
for i = 1 to n/2 {
    print i
    for j = 1 to n, step j = j*2
        print i*j
}
```

Algorithm 3

```
for i = 1 to n/2
    print i
for j = 1 to n, step j = j*2
    print i*j
```

Algorithm 4

```
input x (+ve integer)
while x > 0
    print x
    $x = \lfloor x/5 \rfloor$
```

Analyzing Algorithms

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Analyzing Algorithms

- Algorithm 2

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Analyzing Algorithms

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Analyzing Algorithms

- Algorithm 4

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input x (+ve integer)
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Credits & Book Readings

- Book Readings
  - 2.1, 2.2, 3.1, 3.2
- Credits
  - Prof. Ahmed Eldawy notes
    - http://www.cs.ucr.edu/~eldawy/17WCS141/slides/CS141-1-09-17.pdf
  - Online websites
    - https://commons.wikimedia.org/wiki/File:Exponential.svg