CS141: Intermediate Data Structures and Algorithms

Greedy Algorithms

Amr Magdy

Activity Selection Problem

- Given a set of activities $S = \{a_1, a_2, \ldots, a_n\}$ where each activity i has a start time s_i and a finish time f_i, where $0 \leq s_i < f_i < \infty$.
- An activity a_i happens in the half-open time interval $[s_i, f_i)$.

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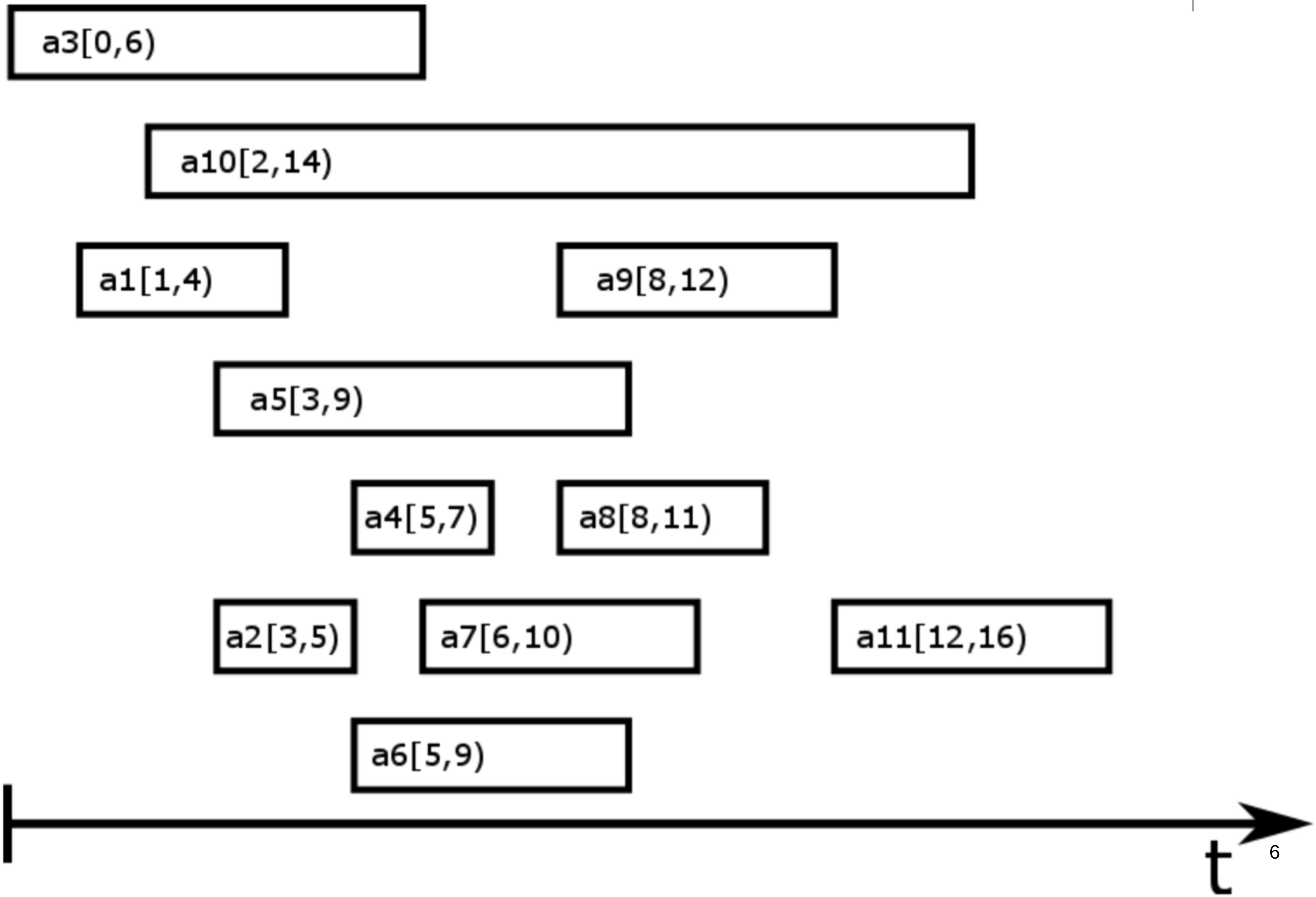
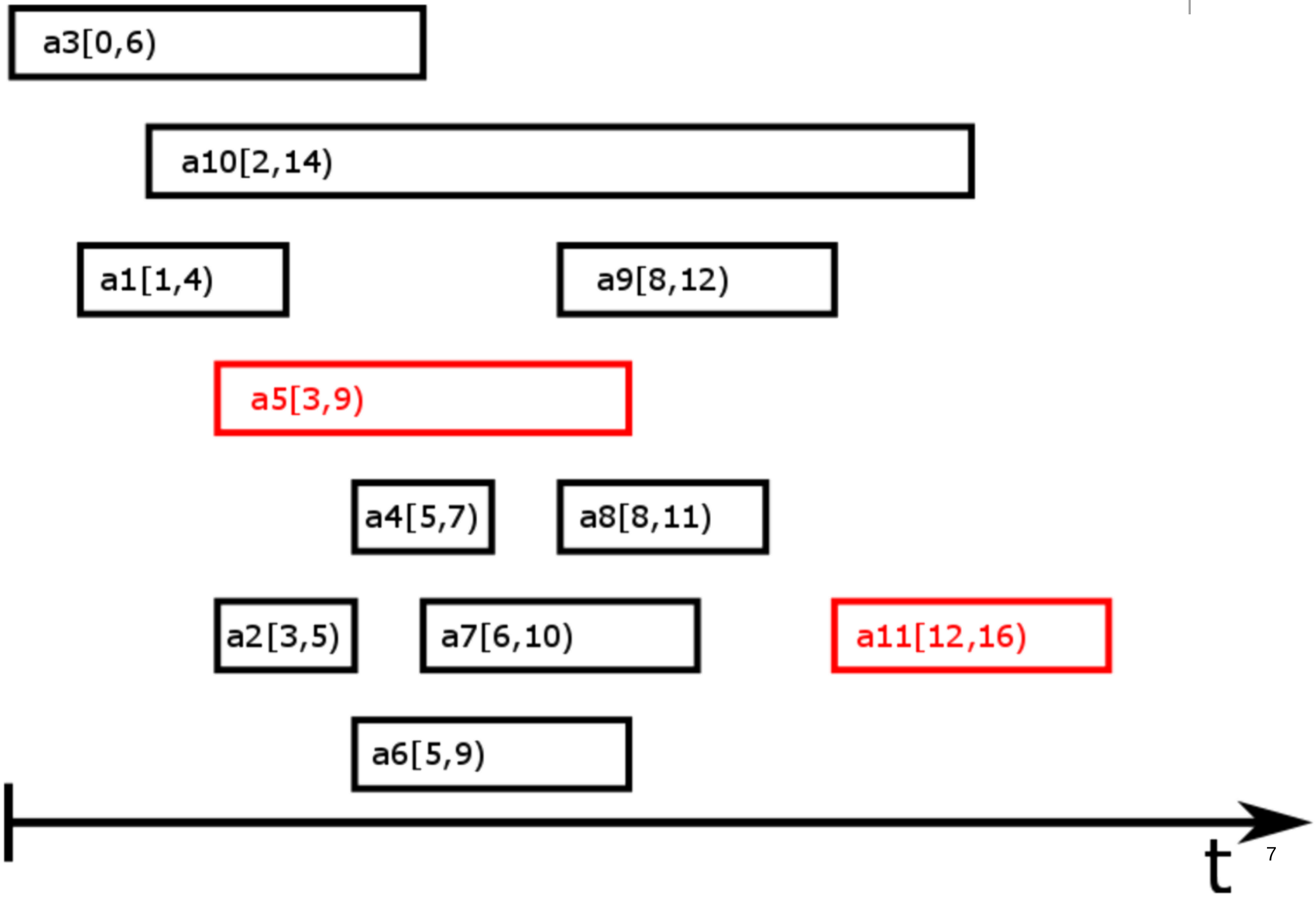
- Activities compete on a single resource, e.g., CPU
- Two activities are said to be **compatible** if they **do not overlap**.
- The problem is to find a **maximum-size compatible subset**, i.e., a one with the maximum number of activities.

Example

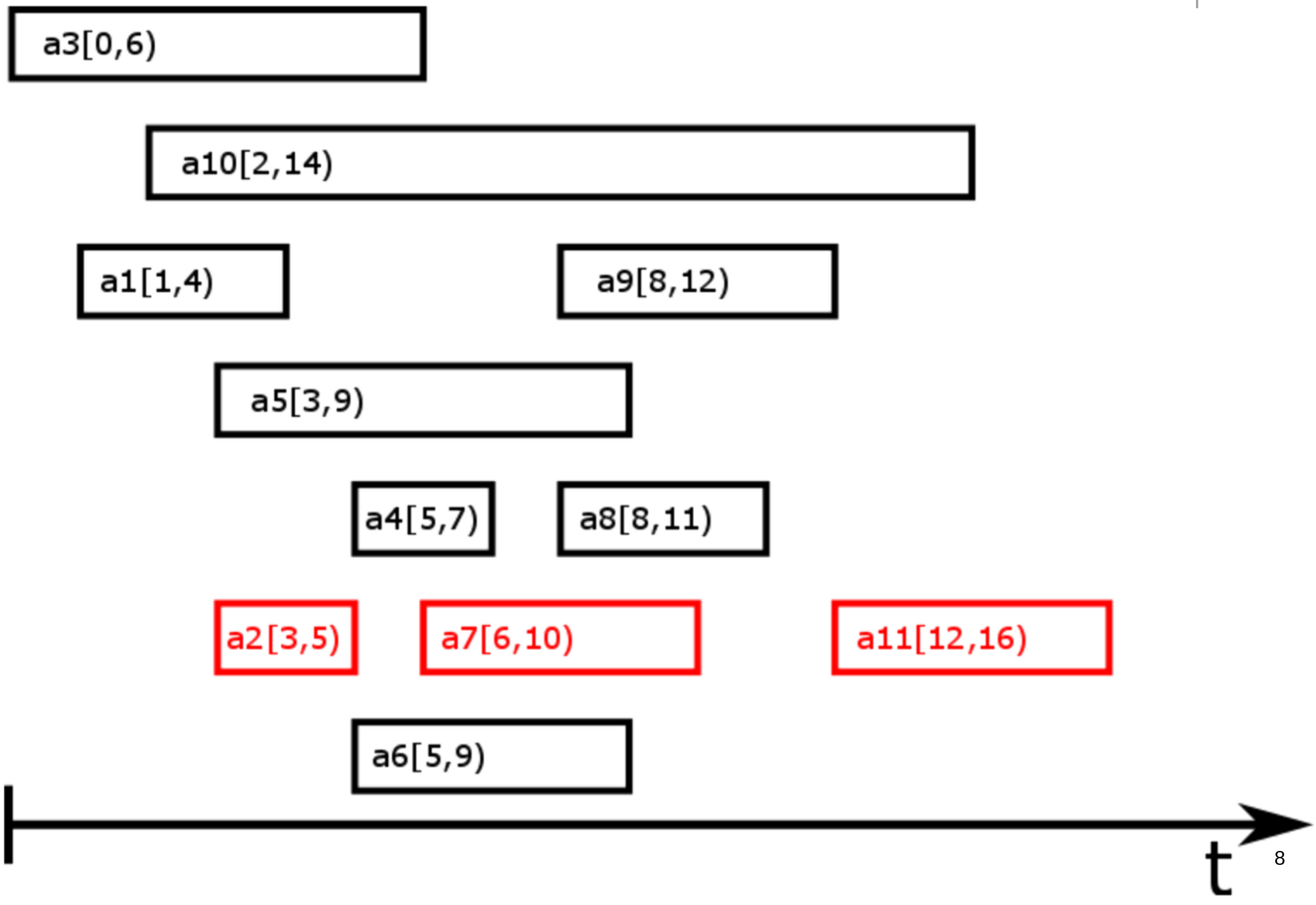


a3[0,6)
a10[2,14)
a1[1,4)
a9[8,12)
a5[3,9)
a4[5,7)
a8[8,11)
a2[3,5)
a7[6,10)
a11[12,16)
a6[5,9)
t

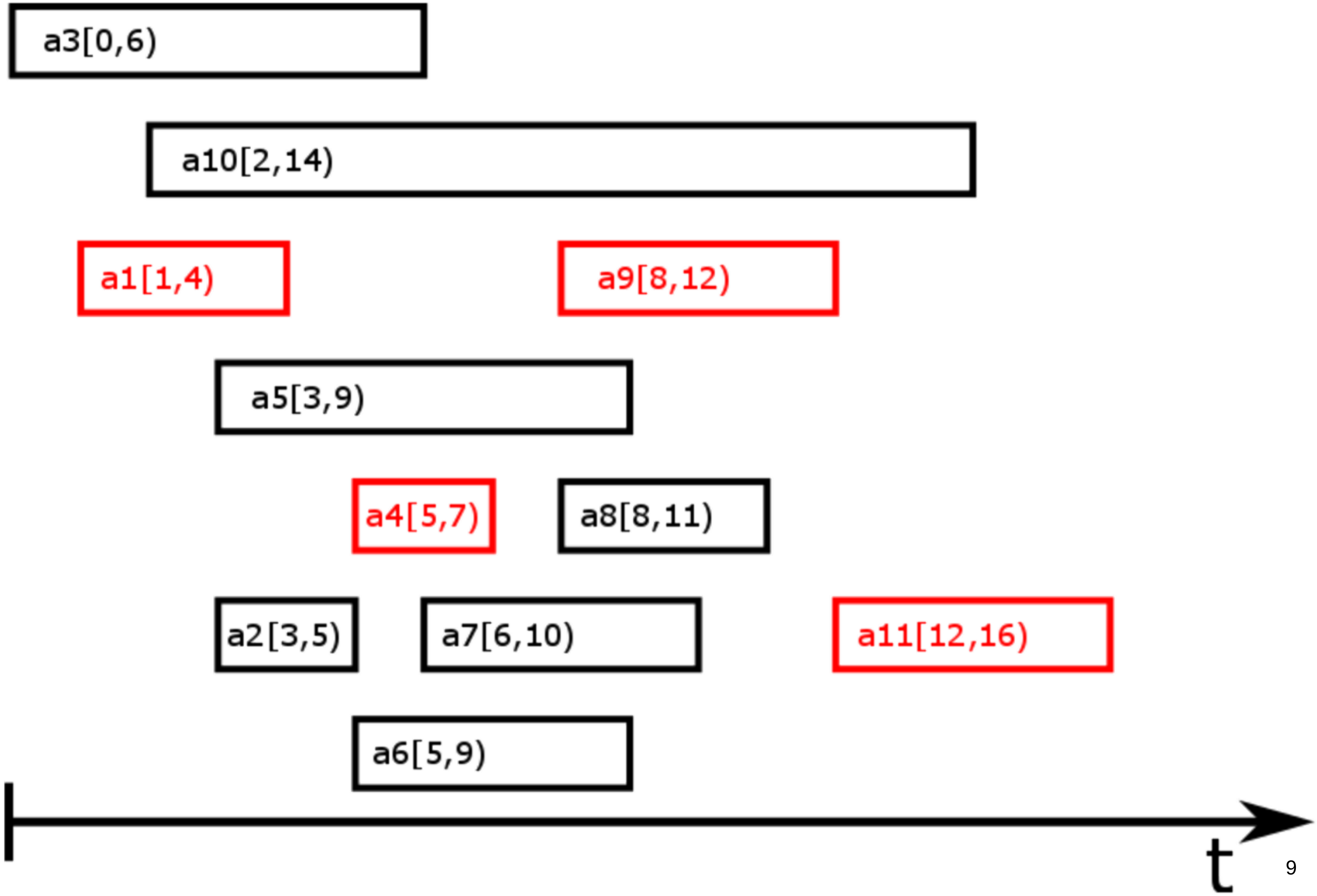
A Compatible Set



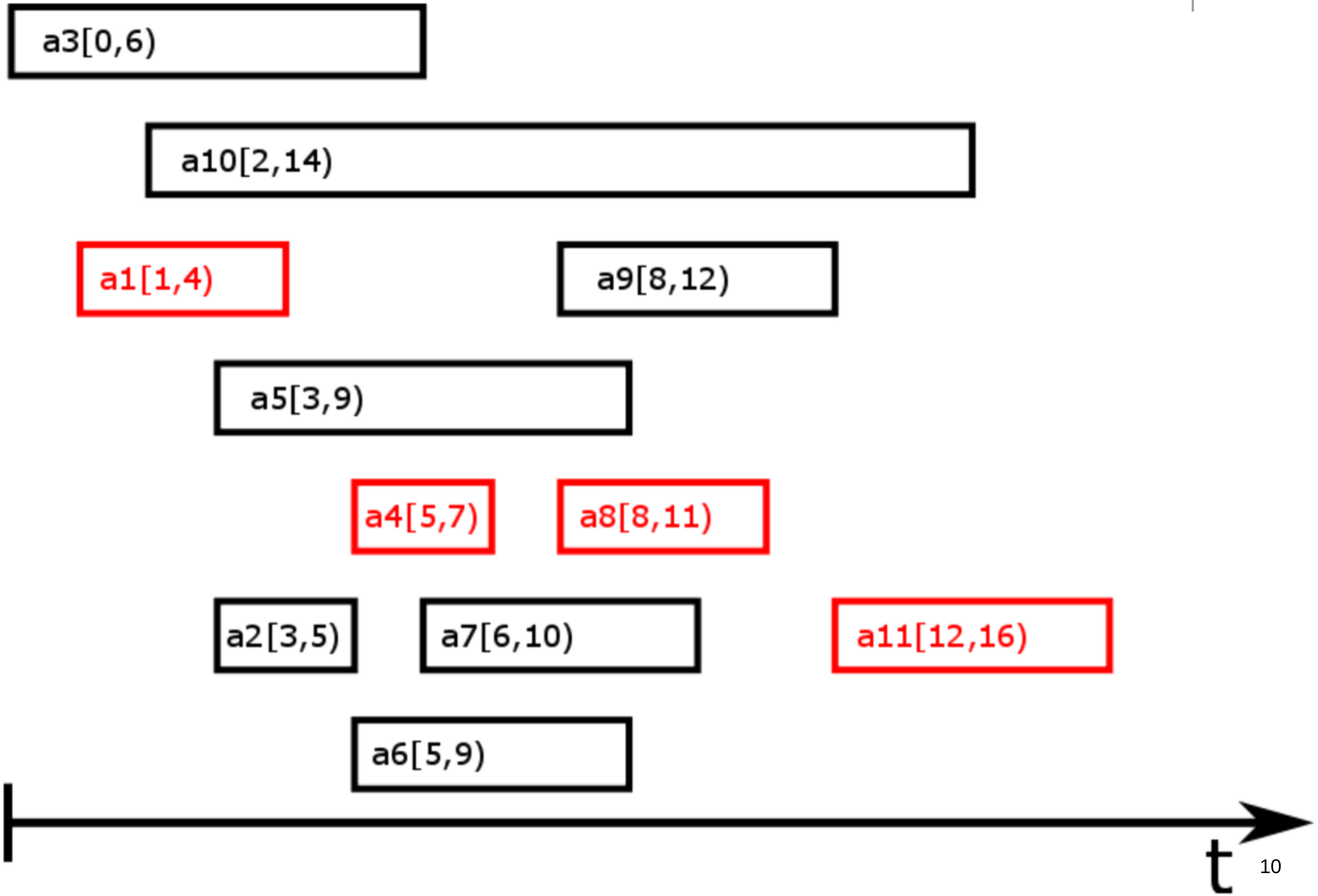
A Better Compatible Set



An Optimal Solution



Another Optimal Solution



Activity Selection Problem

- Solution algorithm?
  - Brute force (naïve): all possible combinations → $O(2^n)$
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- Assume A is an optimal solution for S
  - Is A' = A-{a_i} an optimal solution for S' = S-{a_i and its incompatible activities}?
  - If A' is not an optimal solution, then there an optimal solution A'' for S' so that |A''| > |A'|
  - Then B=A'' U {a_i} is a solution for S, |B|=|A''|+1, |A|=|A'|+1
  - Then |B| > |A|, i.e., |A| is not an optimal solution, **contradiction**
  - Then A' must be an optimal solution for S'

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- Proof by contradiction
  - Assume the opposite of your goal
  - Given that prove a contradiction, then your goal is proved

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- Instead, can we a **greedy** choice?
  - i.e., take the best choice so far, reduce the problem size, and solve a subproblem later
- Greedy choices
  - Longest first
  - Shortest first
  - Earliest start first
  - Earliest finish first
  - …?

Activity Selection Problem

- Greedy choice: earliest finish first
  - Why? It leaves as much resource as possible for other tasks

Activity Selection Problem

- Greedy choice: earliest finish first
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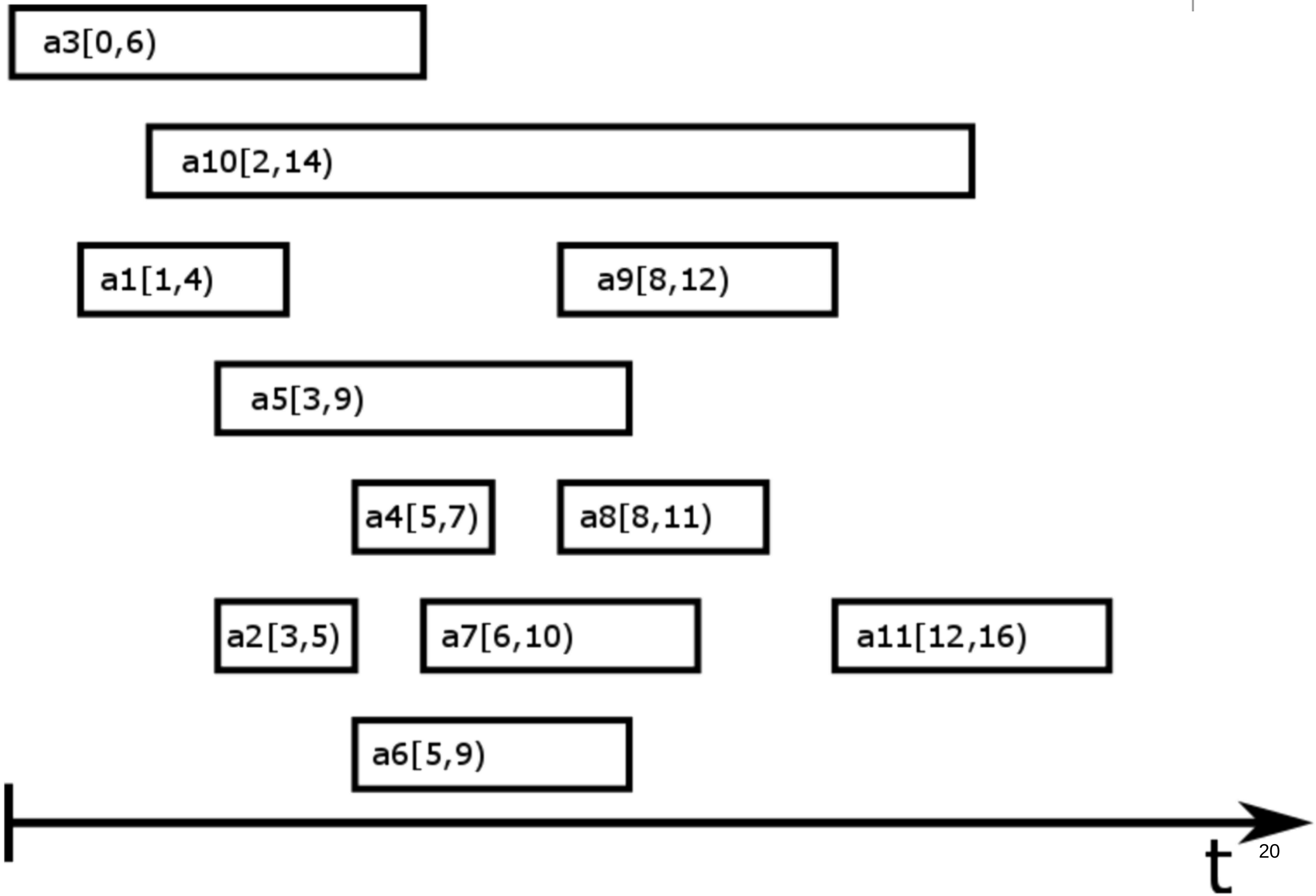
- Solution:
  - Include earliest finish activity a_m in solution A
  - Remove all a_m's incompatible activities
  - Repeat for the remaining earliest finish activity

Activity Selection Problem: Greedy Solution



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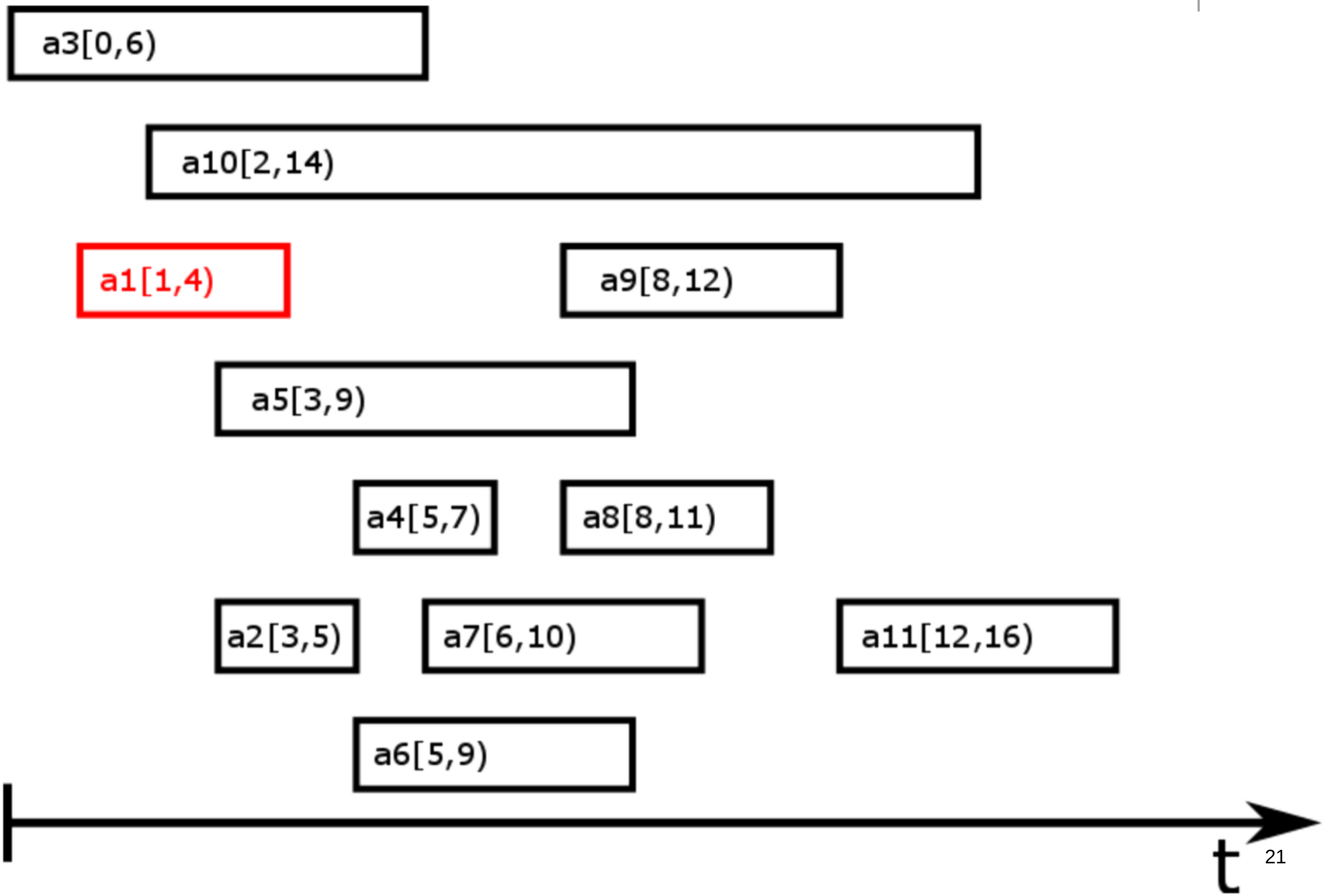

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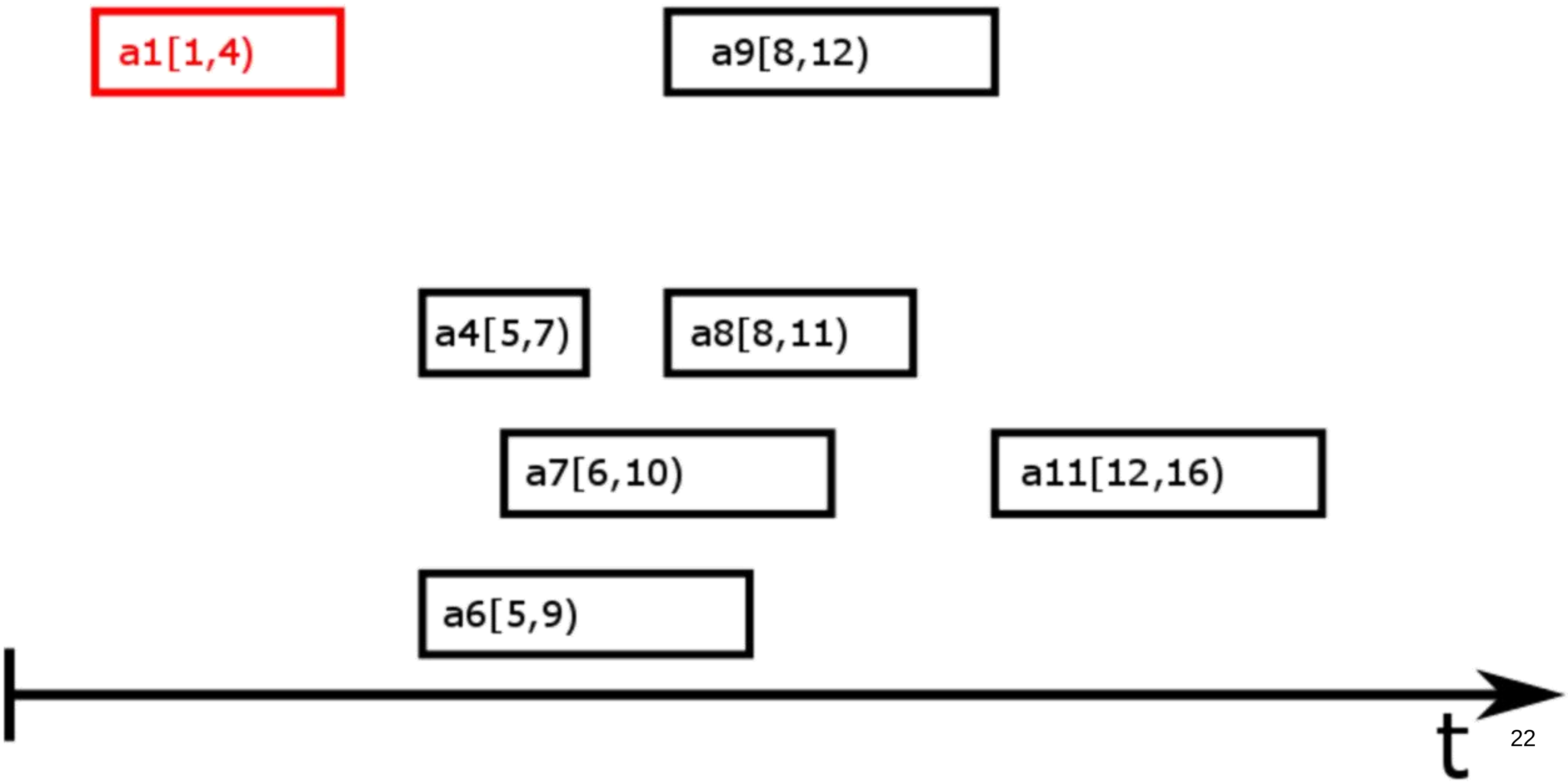

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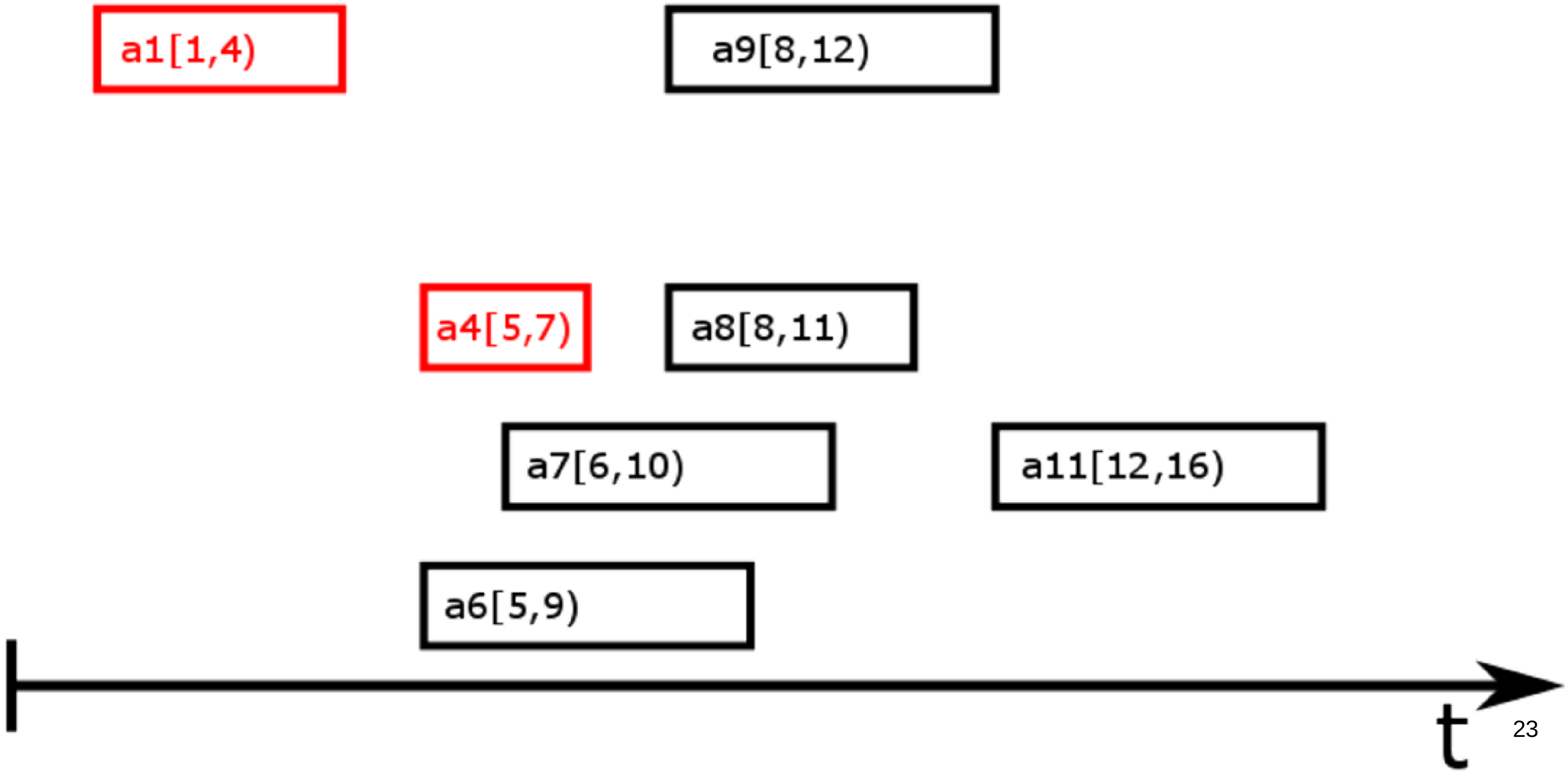

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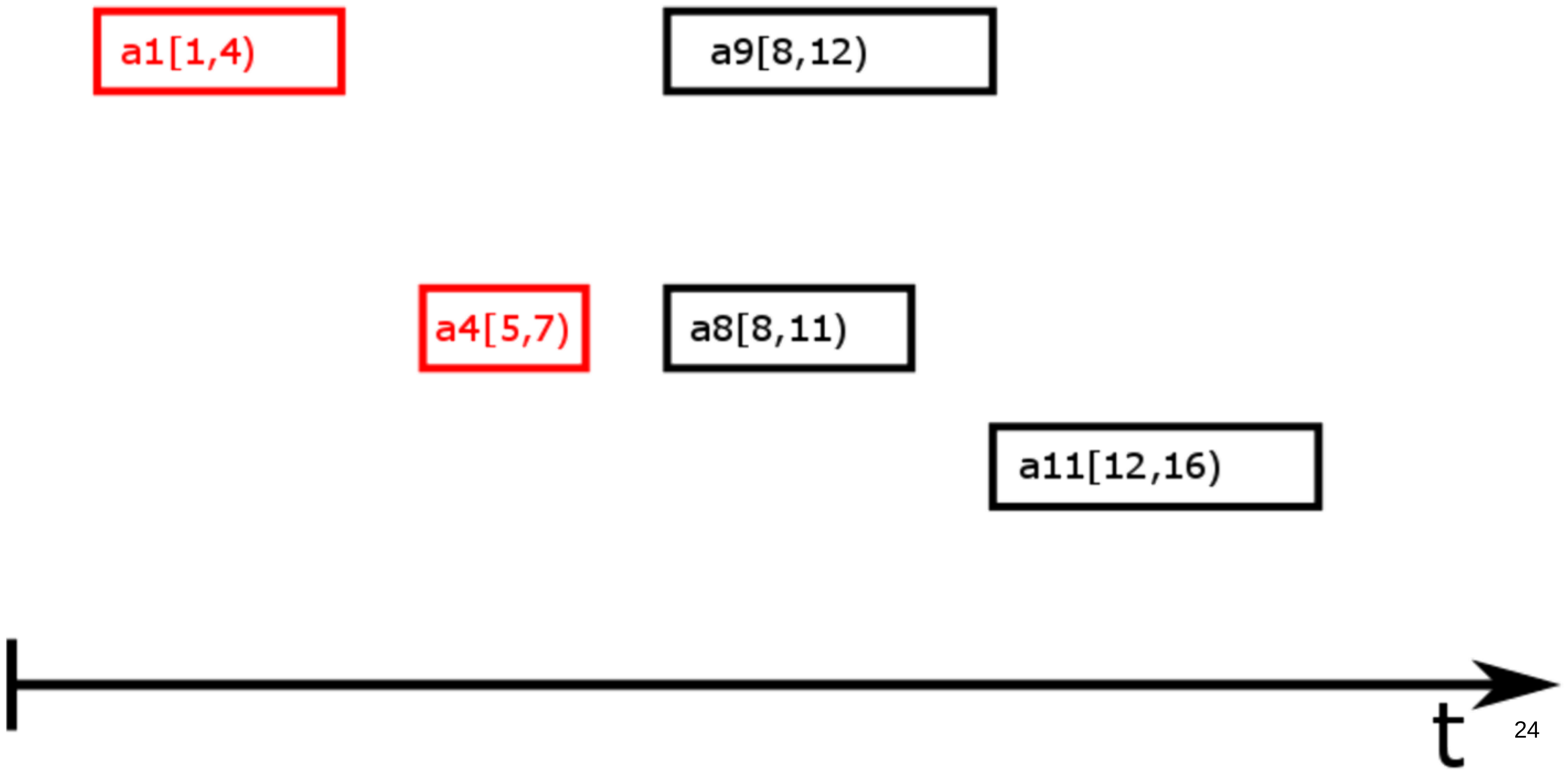

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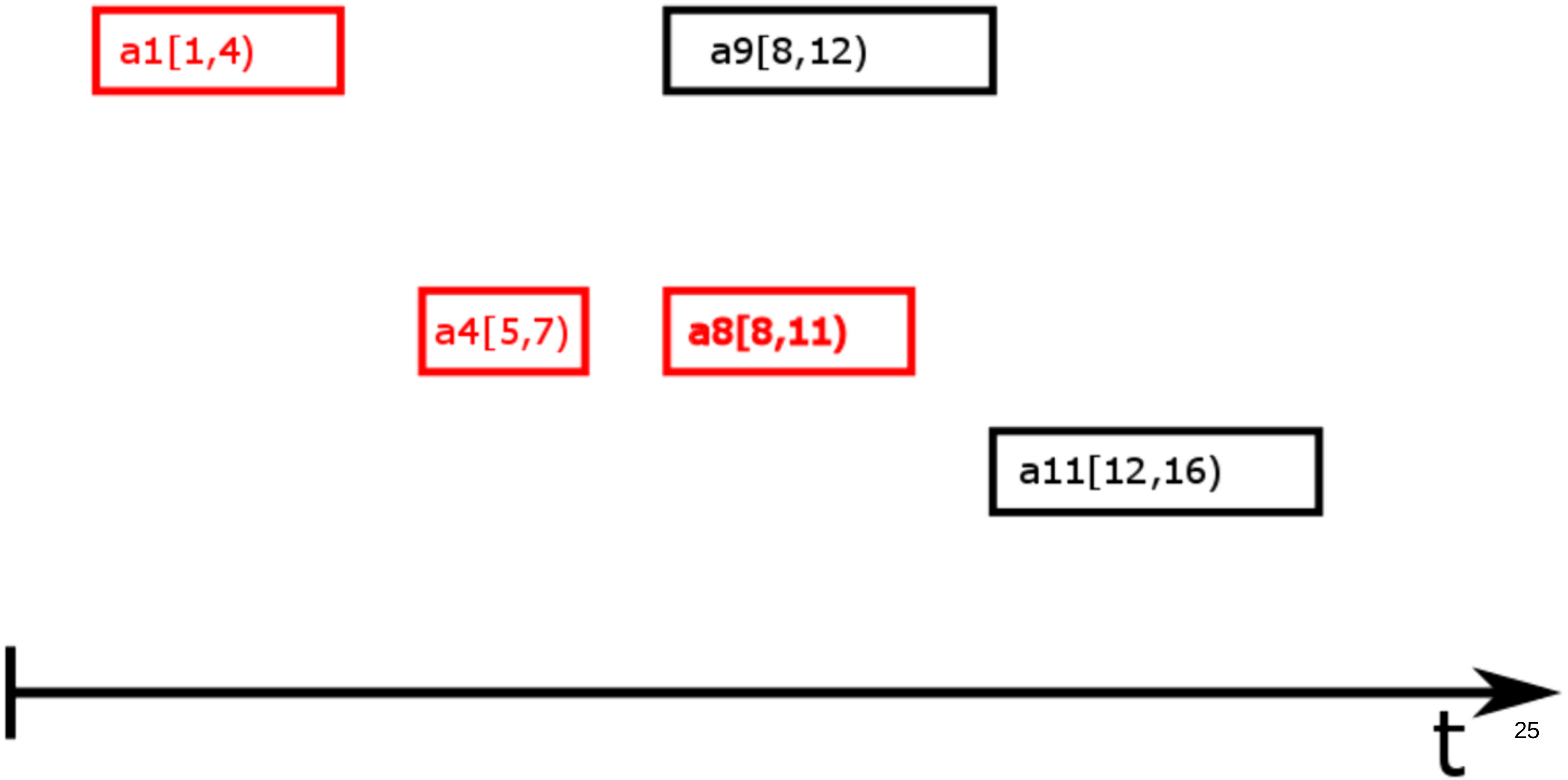

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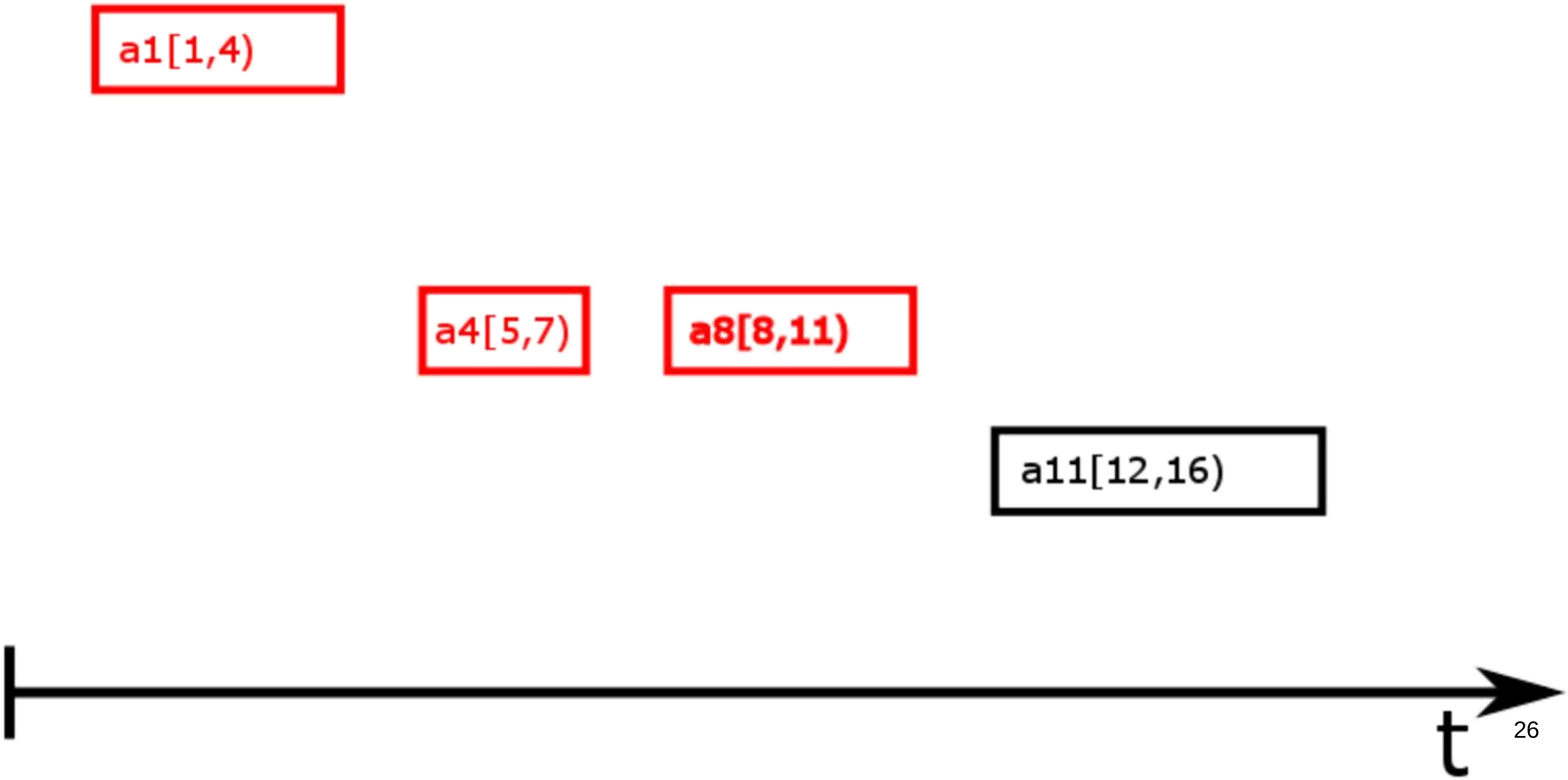

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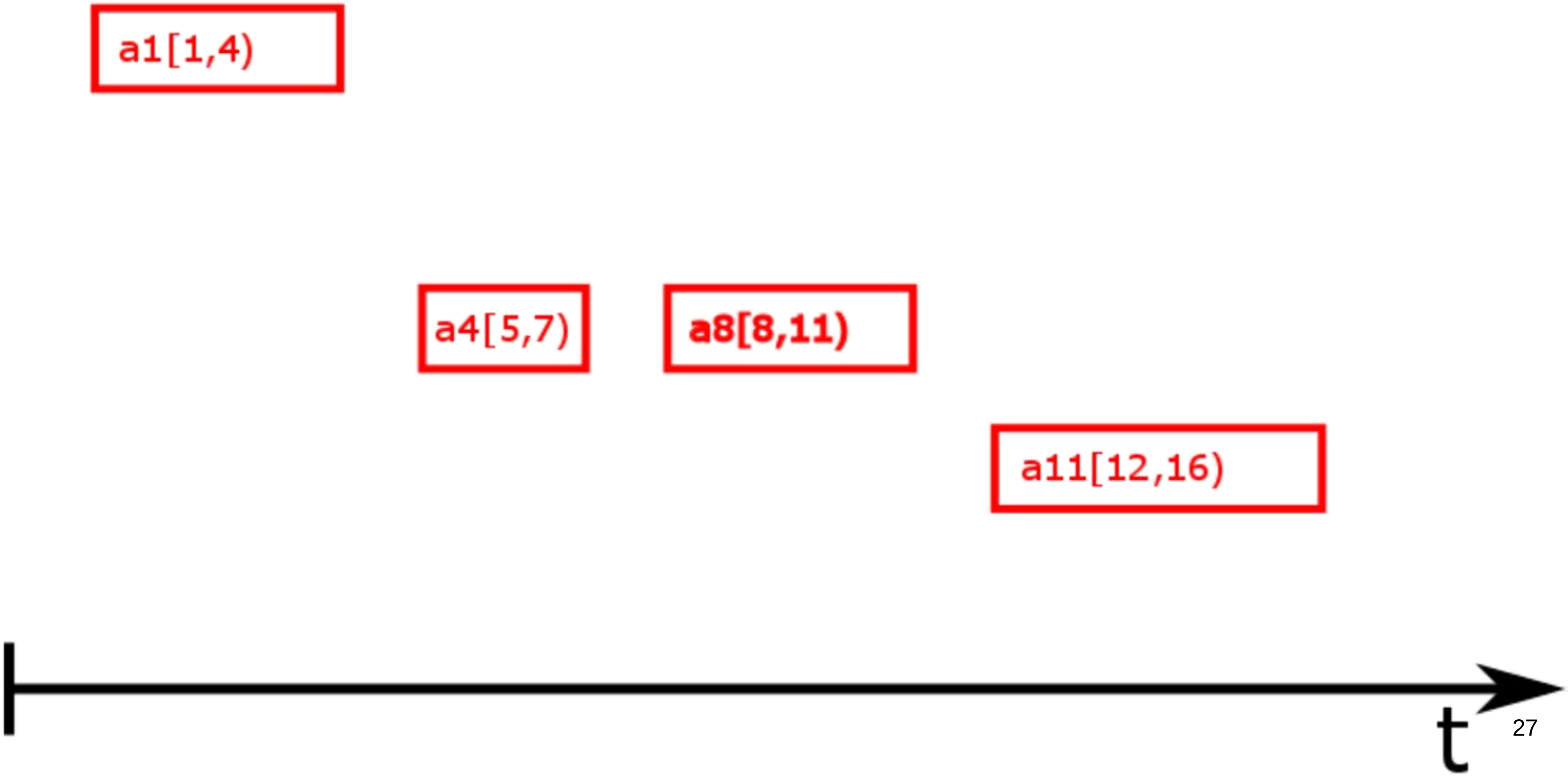

Activity Selection Problem

- Pseudo code?

Activity Selection Problem

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```
findMaxSet(Array a, int n)
{
        - Sort "a" based on earliest finish time
        - result ← {}
        - for i = 1 to n
                validAi = true
                 for j = 1 to result.size
                        if (a[i] is incompatible with result[j])
                                validAi = false
                if (validAi)
                        result ← result U a[i]
        - return result
}
```

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- Greedy choice property
  - Prove that if a_m has the earliest finish time, it must be included in some optimal solution.
- Assume a set S and a solution set A, where $a_m \notin A$
  - Let a_j is the activity with the earliest finish time in A (not in S)
  - Compose another set $A' = A - \{a_j\} \cup \{a_m\}$
  - A' still have all activities disjoint (as a_m has the global earliest finish time and A activities are already disjoint), and $|A'|=|A|$
  - Then A' is an optimal solution
  - Then a_m is always included in an optimal solution

Elements of a Greedy Algorithm

1. Optimal Substructure
2. Greedy Choice Property

Greedy vs. Dynamic Programming

- Solving the bigger problem include
  One choice (greedy) vs Multiple possible choices

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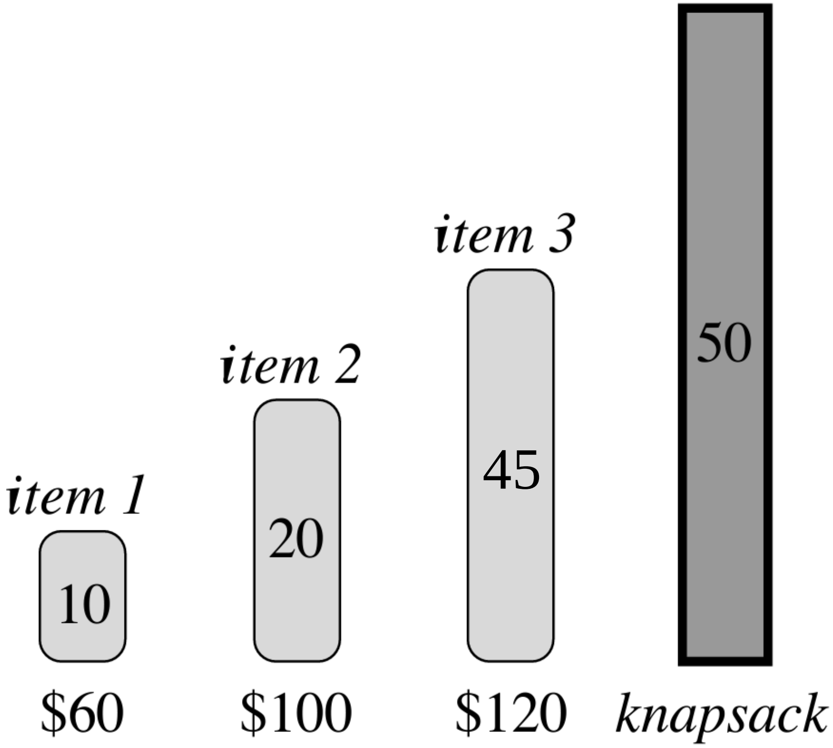
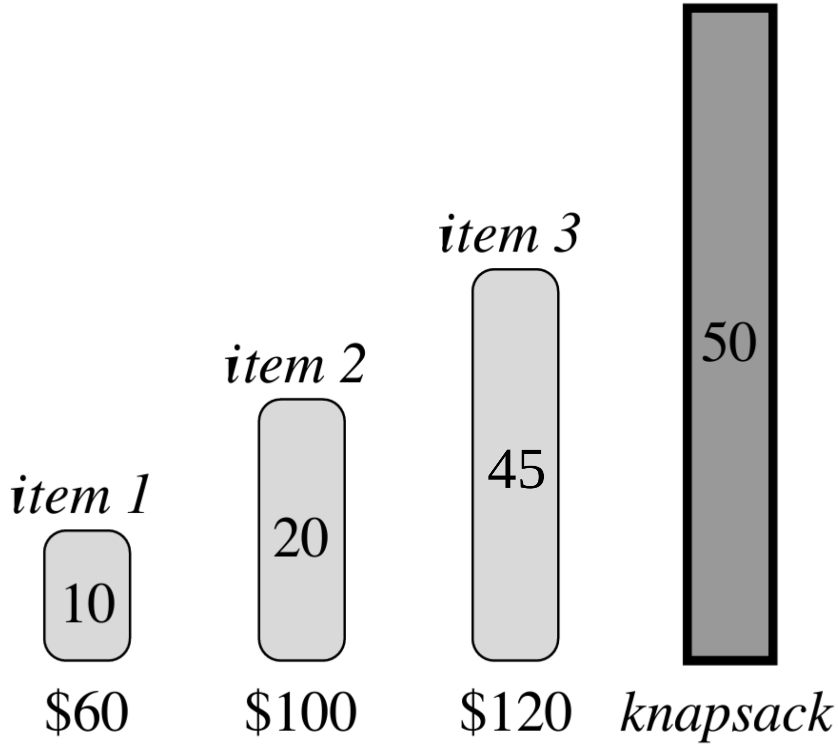
- Elements:

Greedy	DM
Optimal substructure	Optimal substructure
Greedy choice property	Overlapping subproblems

Knapsack Problem

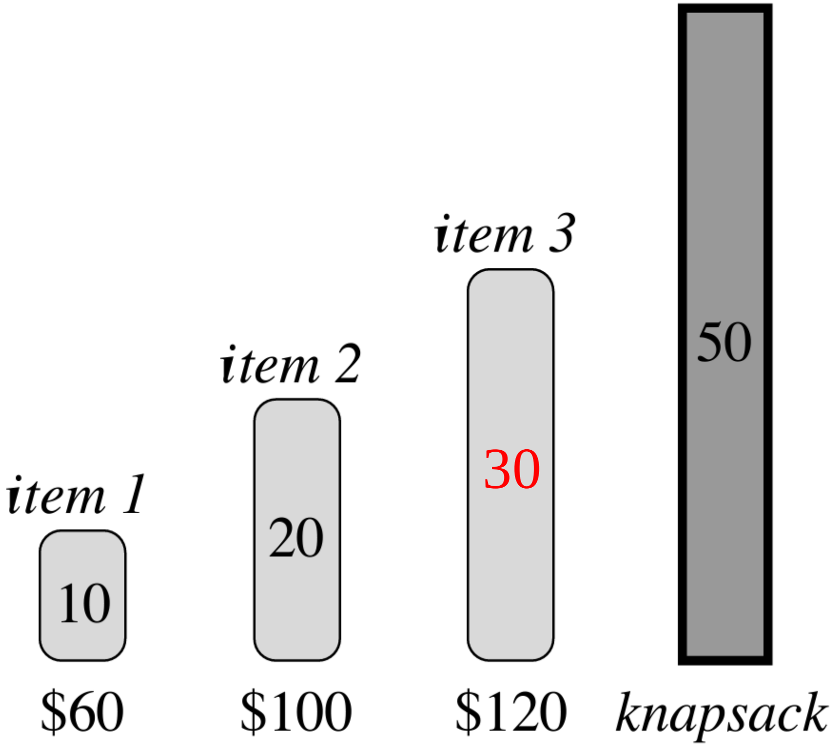


Knapsack Problem



- 0-1 Knapsack: Each item either included or not
- Greedy choices:
  - Take the most valuable → Does not lead to optimal solution
  - Take the most valuable per unit → Works in this example

Knapsack Problem



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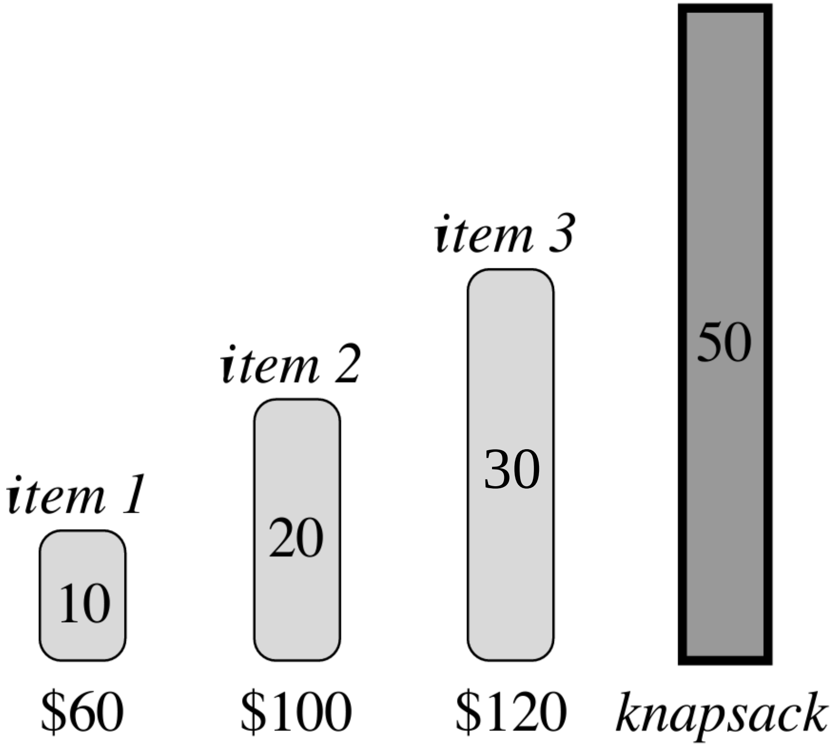

- Fractional Knapsack: Part of items can be included

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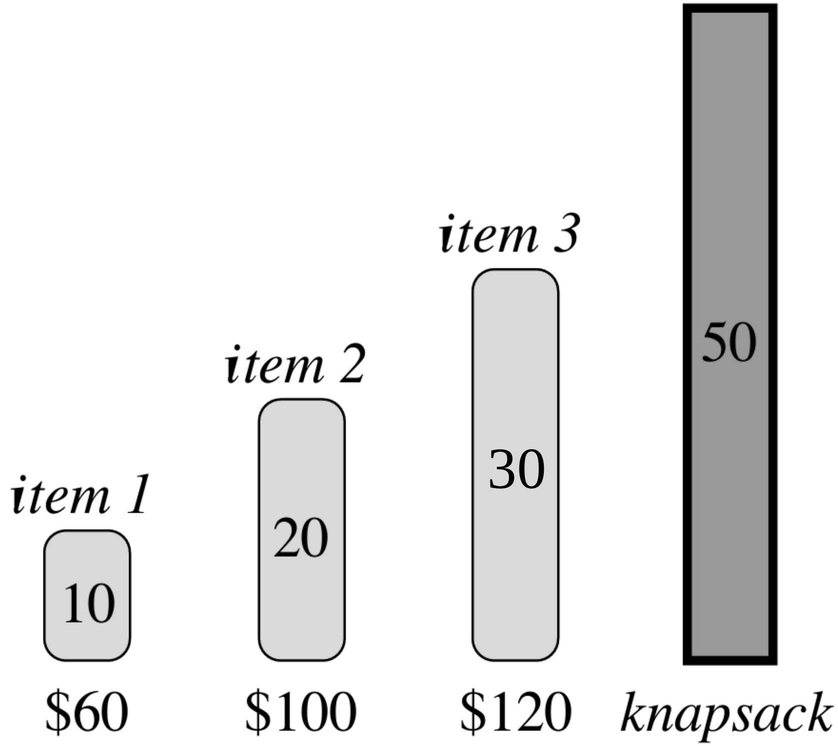

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Fractional Knapsack Problem

- Greedy choice property: take the most valuable per weight unit

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- Greedy choice property: take the most valuable per weight unit
- Proof of optimality:
  - Given the set S ordered by the value-per-weight, taking as much as possible x_j from the item j with the highest value-per-weight will lead to an optimal solution X
  - Assume we have another optimal solution $X`$ where we take less amount of item $j$, say $x_j` < x_j$.
  - Since $x_j` < x_j$, there must be another item $k$ which was taken with a higher amount in $X`$, i.e., $x_k` > x_k$.
  - We create another solution $X``$ by doing the following changes in $X`$
    - Reduce the amount of item k by a value z and increase the amount of item j by a value z
    - The value of the new solution $V`` = V` + z\ v_j/w_j - z\ v_k/w_k$ $= V` + z\ (v_j/w_j - v_k/w_k) \rightarrow v_j/w_j - v_k/w_k \geq 0 \rightarrow V`` \geq V`$

Fractional Knapsack Problem

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- Given the problem S with an optimal solution X with value V, we want to prove that the solution X` = $X - x_j$ is optimal to the problem $S$` = S - {j} and the knapsack capacity W` = $W - x_j$
- Proof by contradiction
  - Assume that X` is not optimal to $S$`
  - There is another solution X`` to $S$` that has a higher total value $V$`` > V`
  - Then X`` U {$x_j$} is a solution to $S$ with value $V$``+ x_j> V`+ x_j > V
  - Contradiction as V is the optimal value

Fractional Knapsack Problem

```
Fknapsack (W, S, v's, w's) {
        - Sort S based on vi/wi value
        - rw = W
        - result = { }
        - for each si in S
                if(wi <= rw)
                        result = result U si
                        rw = rw-wi
                else
                        result = result U rw/wi * si
                        rw = 0
        - return result
}
```

Huffman Codes

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

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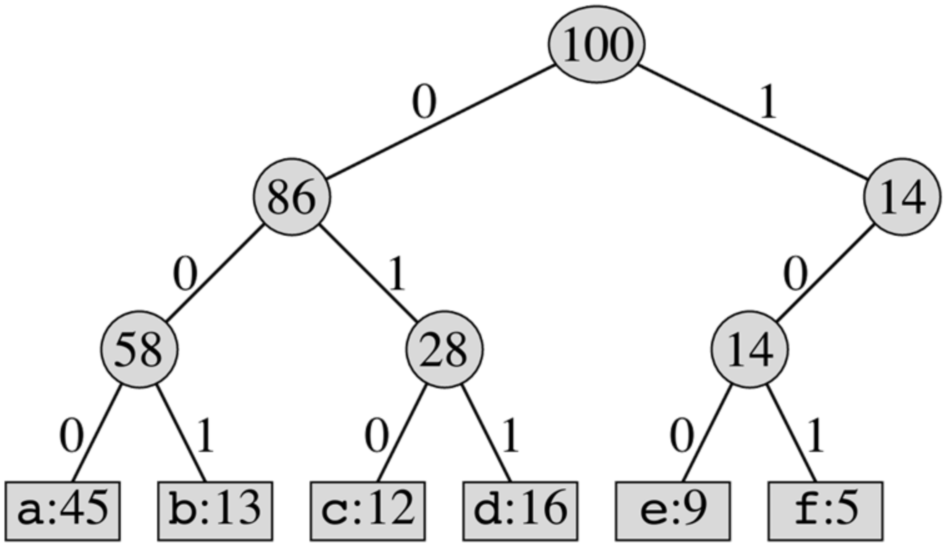
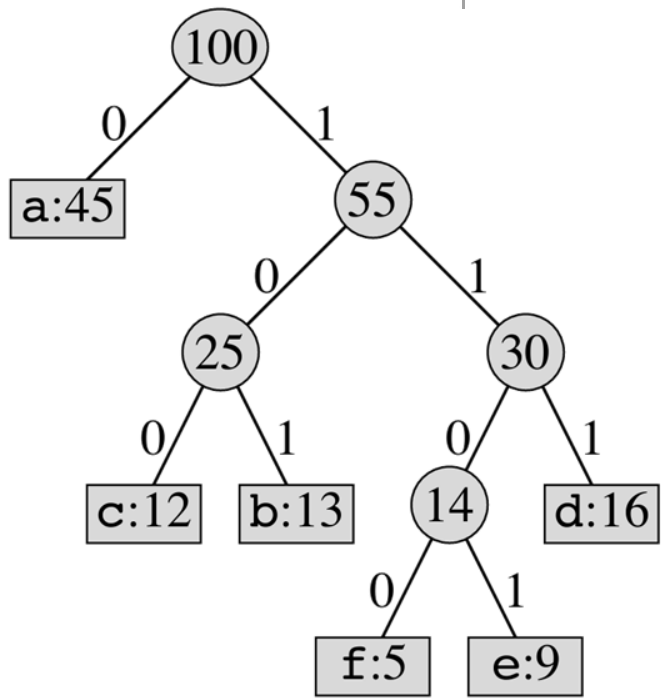
- Example: Message ‘JAVA’ a = “0”, j = “11”, v = “10”
  Encoded message “110100” Decoding “110100”
- In the table:
  Encoding with fixed-length needs 300K bits
  Encoding with variable-length needs 224K bits

Huffman Codes

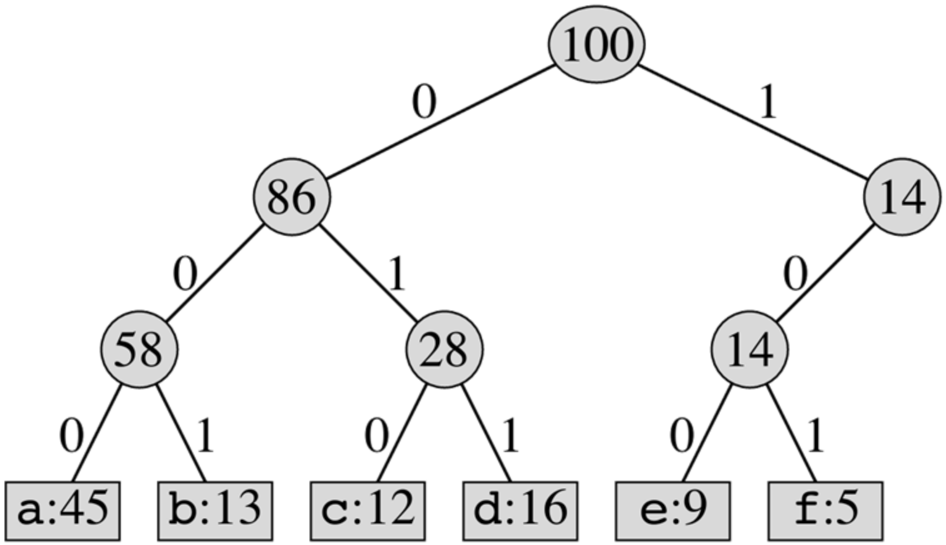


Fixed-length tree



Variable-length tree

Huffman Codes



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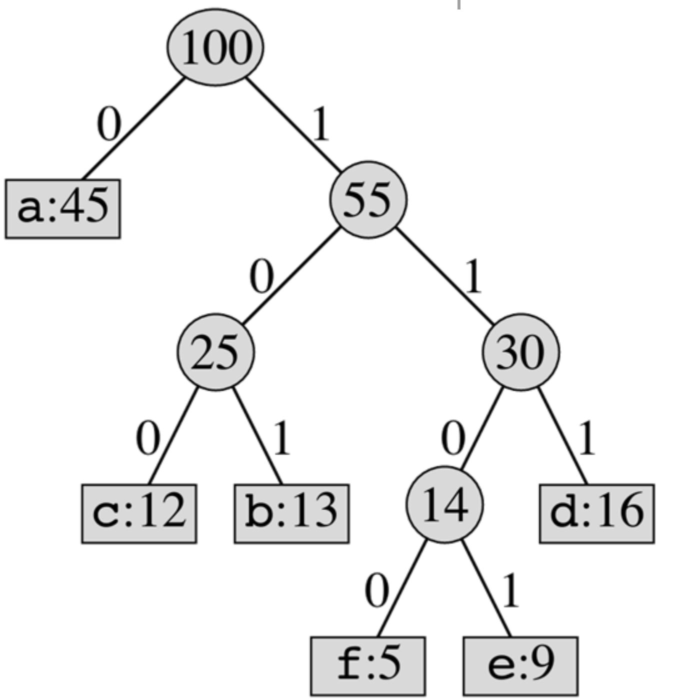


Variable-length tree

We need an algorithm to build the optimal variable-length tree

Huffman Codes: Tree Construction

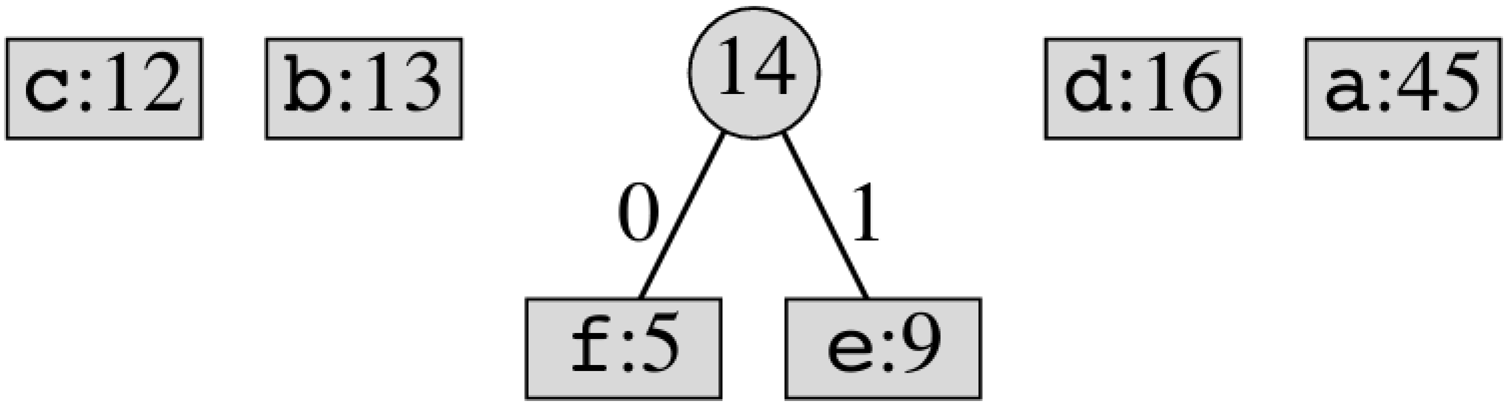
$\text{HUFFMAN}(C)$

$n = |C|$
$Q = C$
for $i = 1$ **to** $n - 1$
    allocate a new node z
    $z.left = x = \text{EXTRACT-MIN}(Q)$
    $z.right = y = \text{EXTRACT-MIN}(Q)$
    $z.freq = x.freq + y.freq$
    $\text{INSERT}(Q, z)$
return $\text{EXTRACT-MIN}(Q)$ // return the root of the tree

Huffman Codes: Tree Construction

f:5	e:9	c:12	b:13	d:16	a:45

Huffman Codes: Tree Construction



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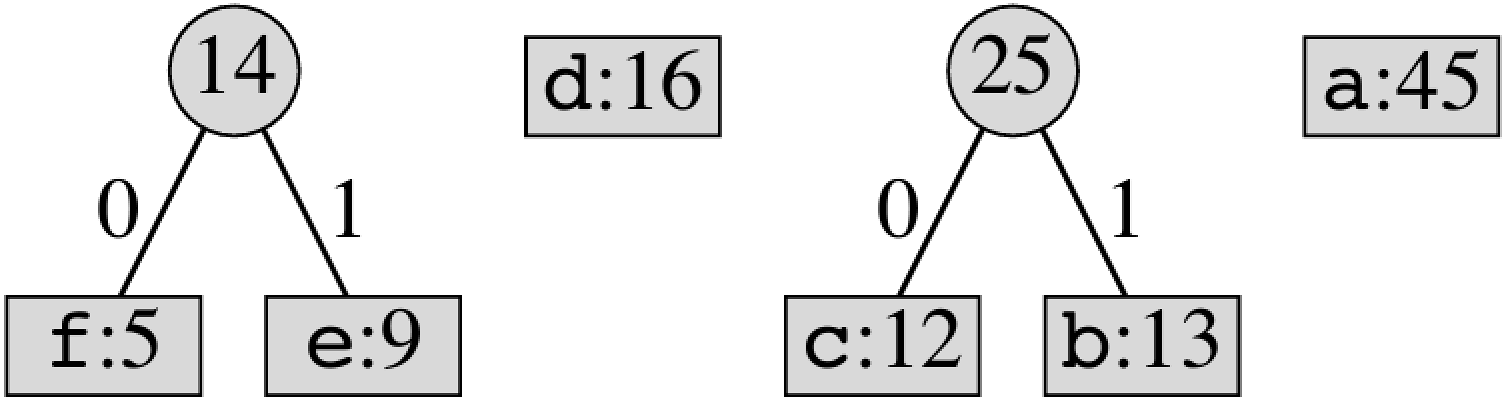

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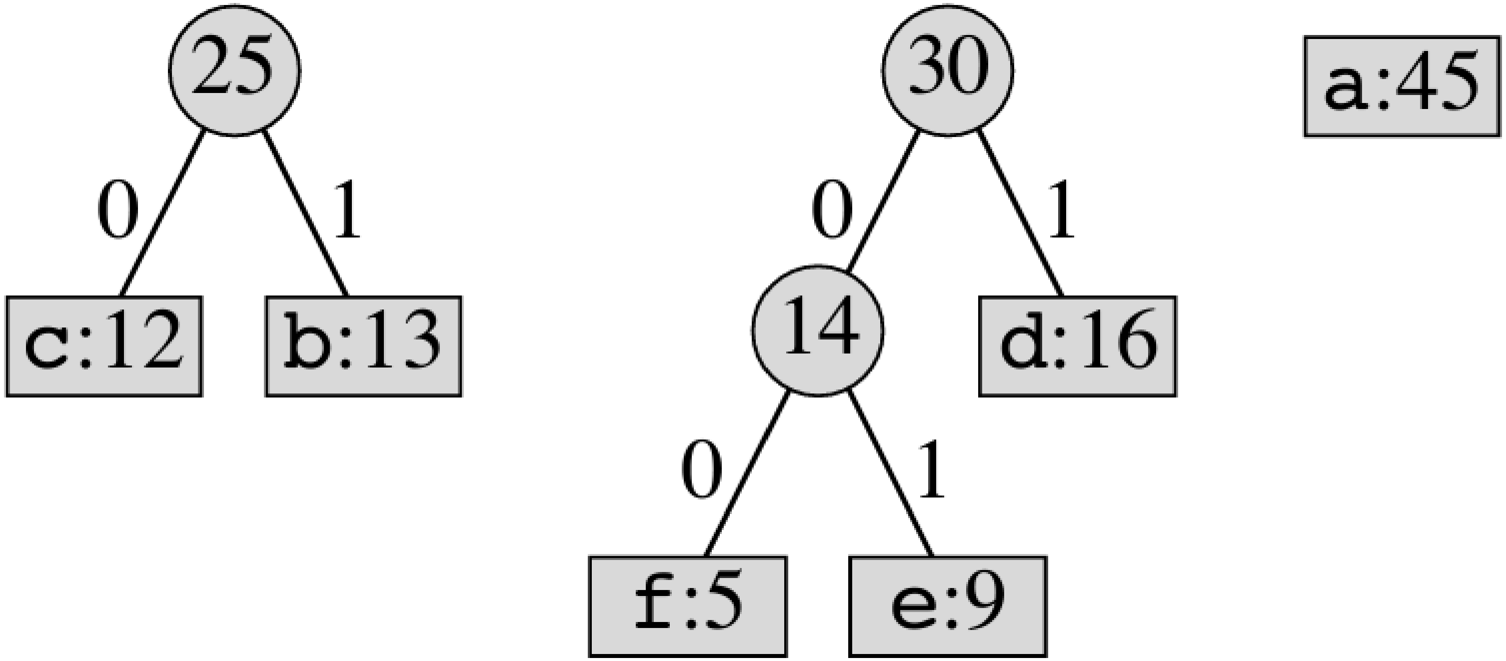

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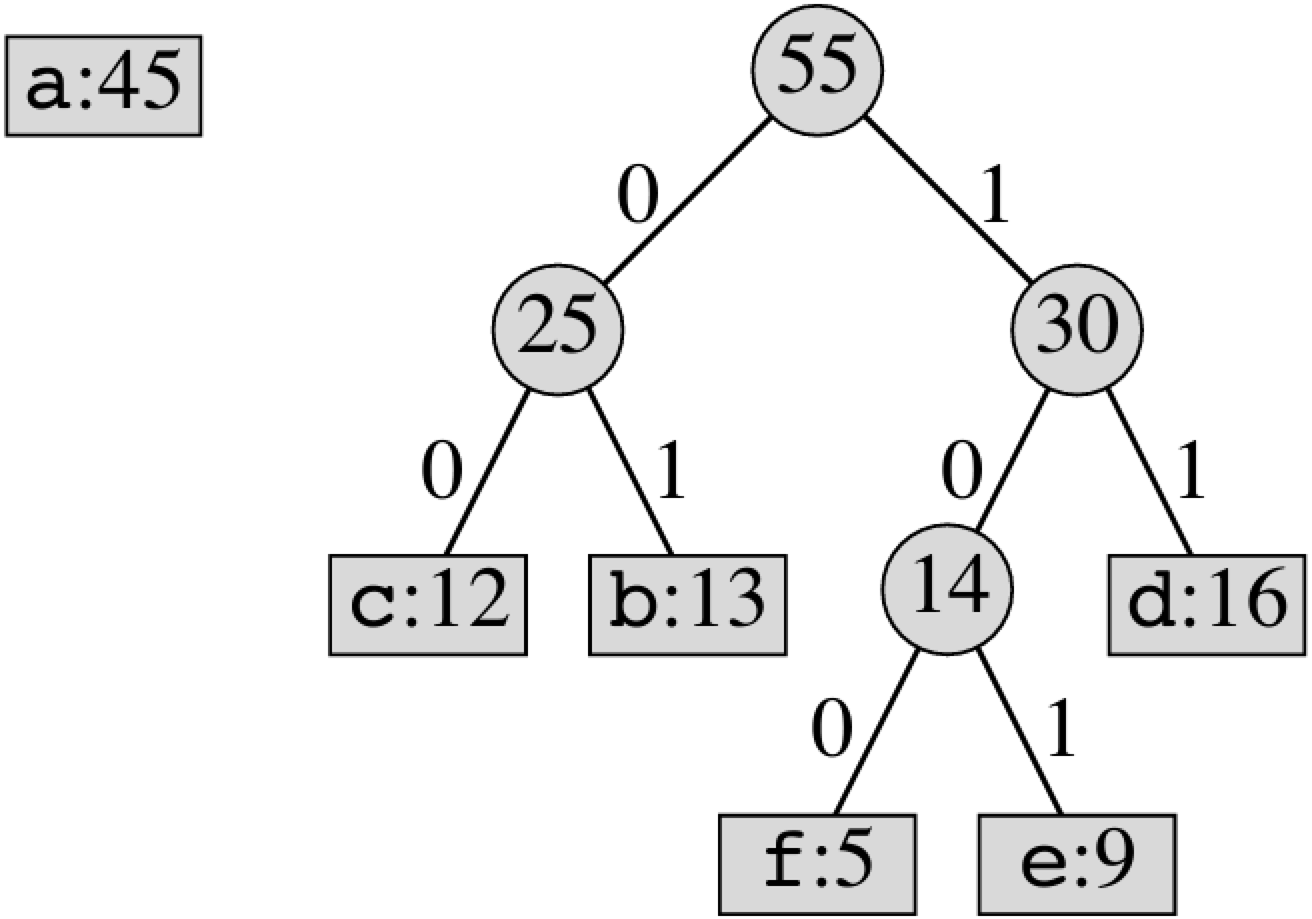

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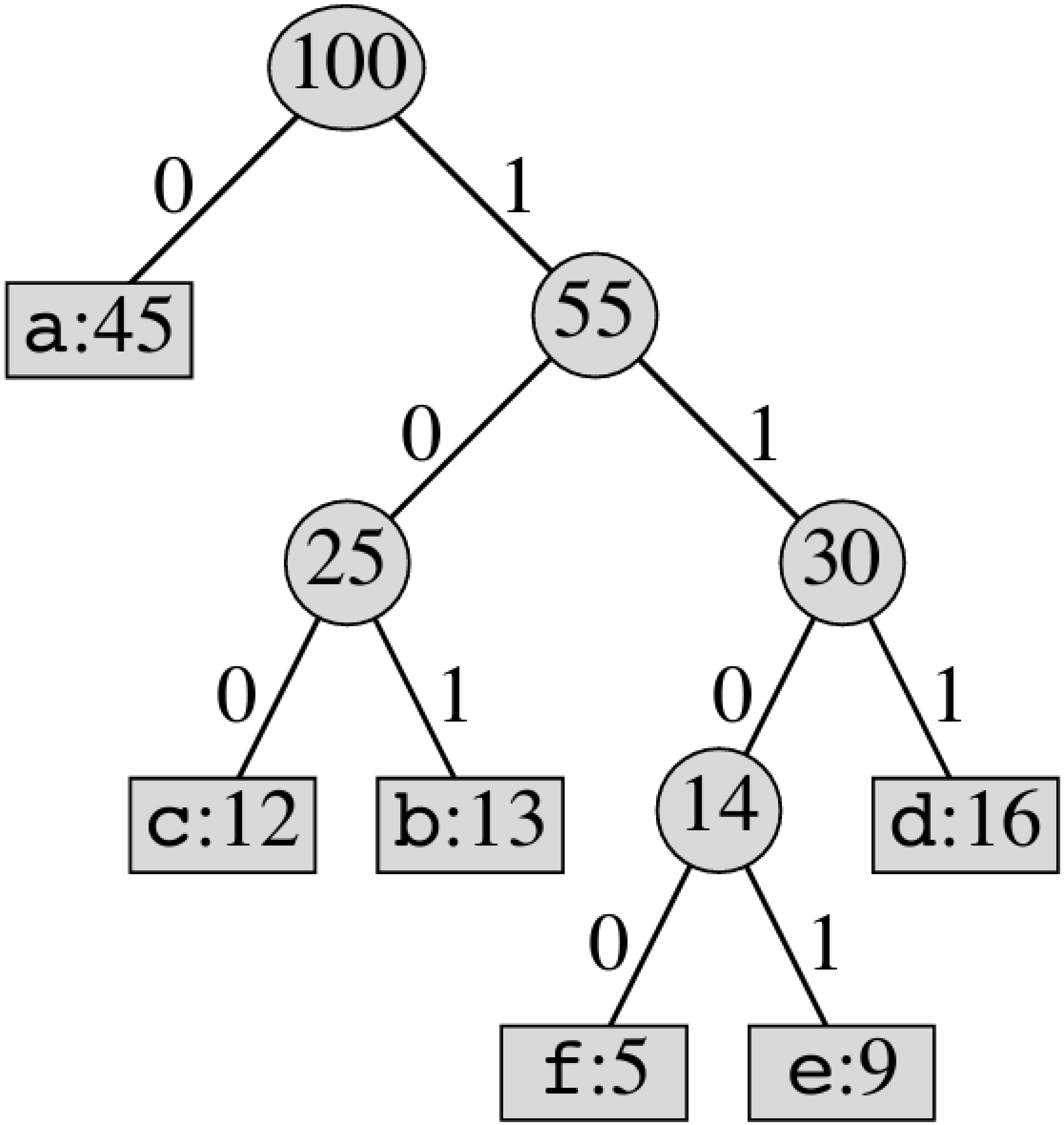

Huffman Codes

- Details of optimal substructure and greedy choice property in the text book

Book Readings and Credits

- Book Readings:
  - 16.1 – 16.3
- Credits to:
  - Prof. Ahmed Eldawy notes