CS141: Intermediate Data Structures and Algorithms

Analysis of Algorithms

Amr Magdy

Analyzing Algorithms

1. Algorithm Correctness
    a. Termination
    b. Produces the correct output for all possible input.

2. Algorithm Performance
    a. Either runtime analysis,
    b. or storage (memory) space analysis
    c. or both

Algorithm Correctness

- Sorting problem
  - Input: an array A of n numbers
  - Output: the same array in ascending sorted order (smallest number in A[1] and largest in A[n])

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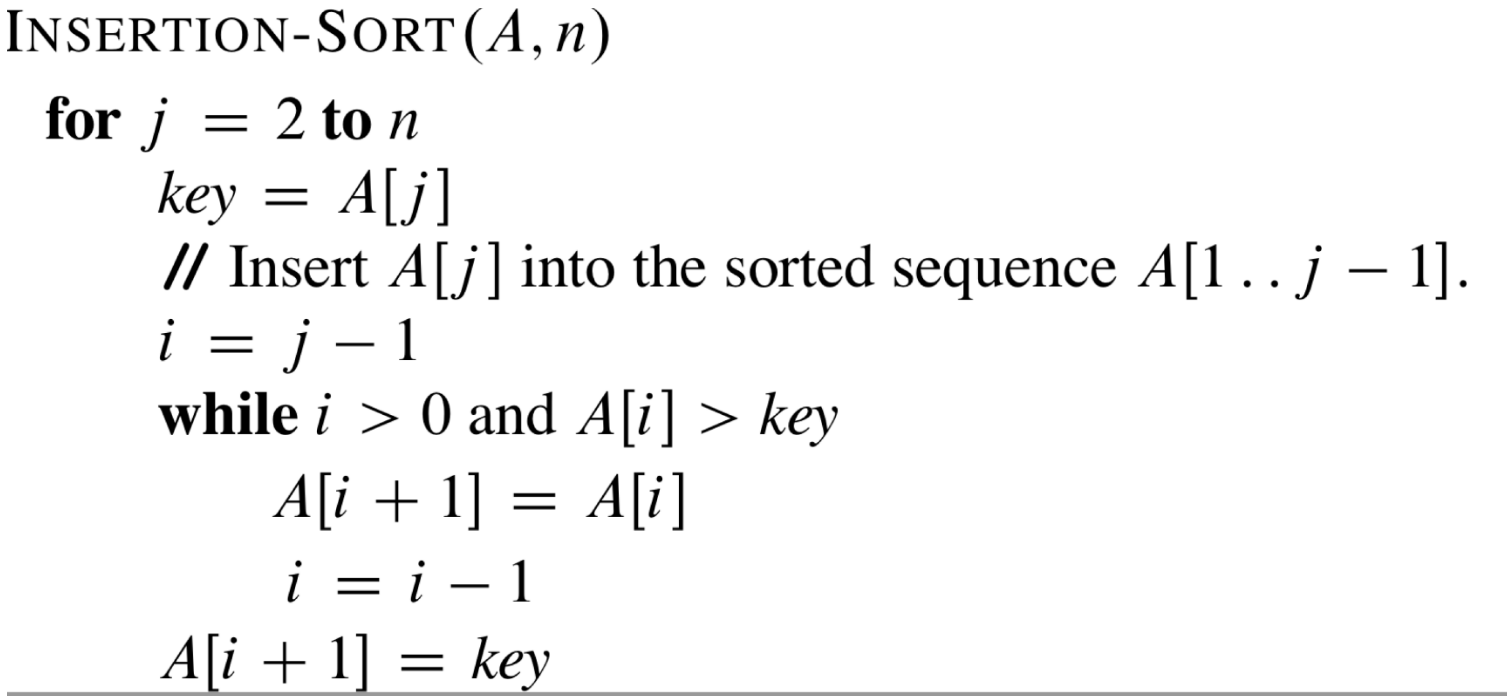
- Insertion Sort



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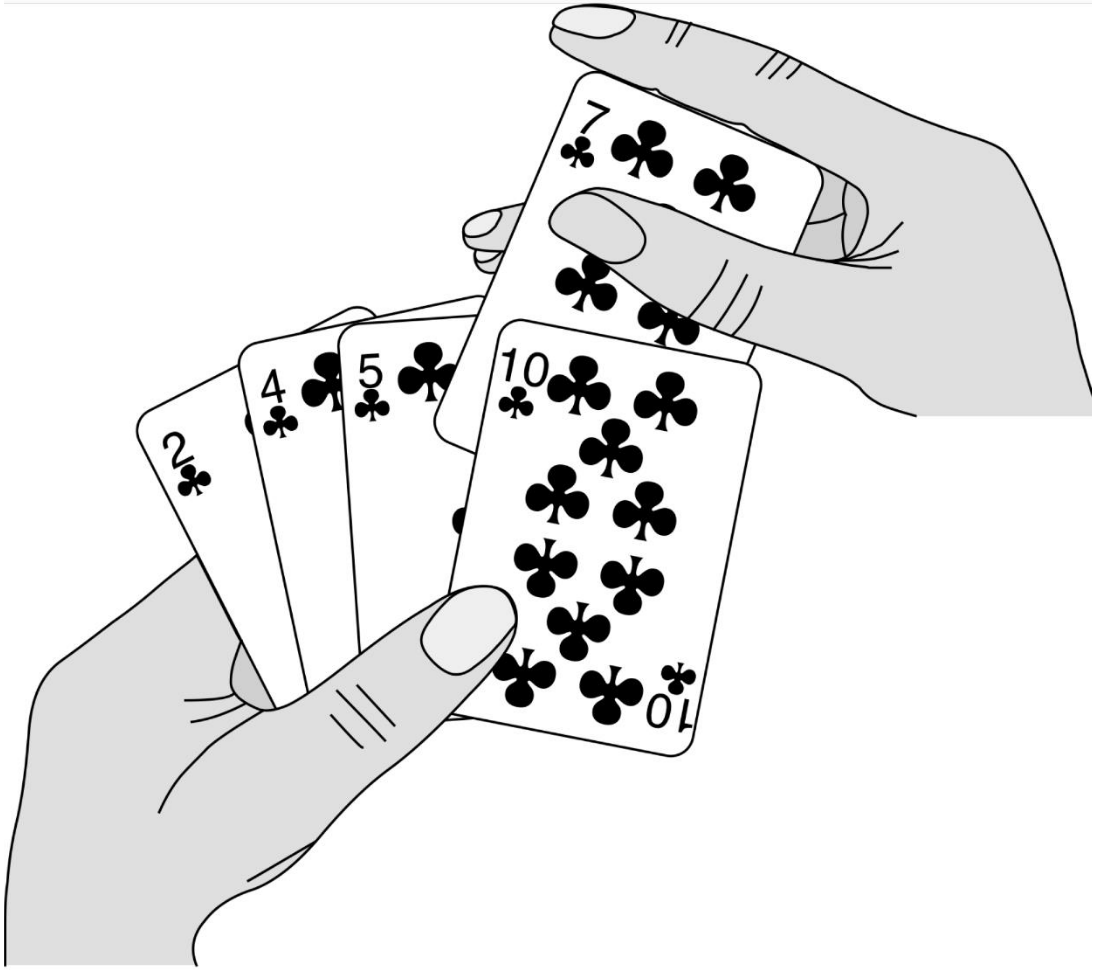
Algorithm Correctness

- How does insertion sort work?



Algorithm Correctness

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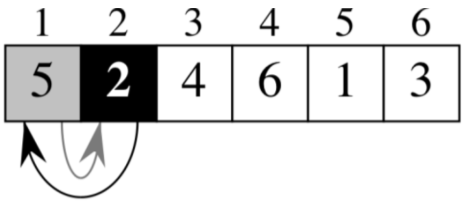

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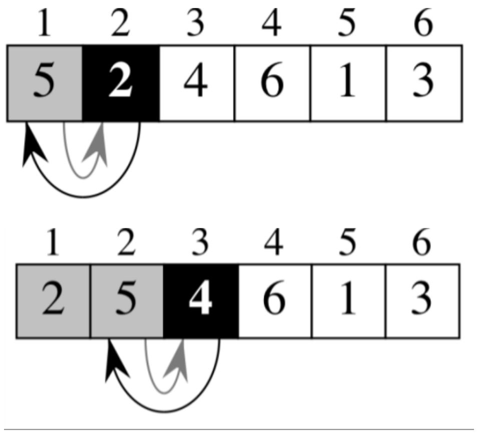

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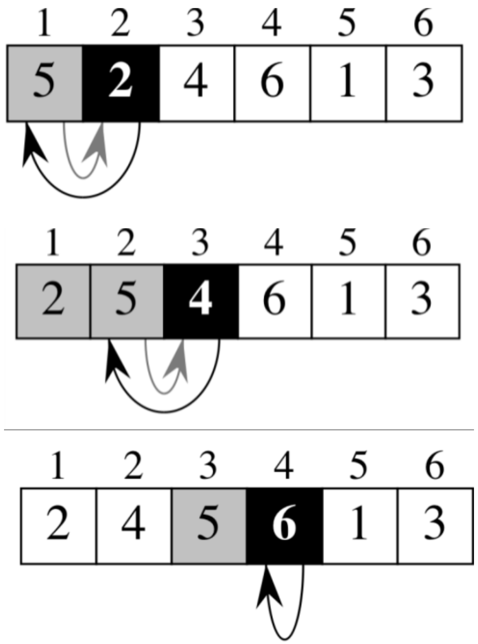

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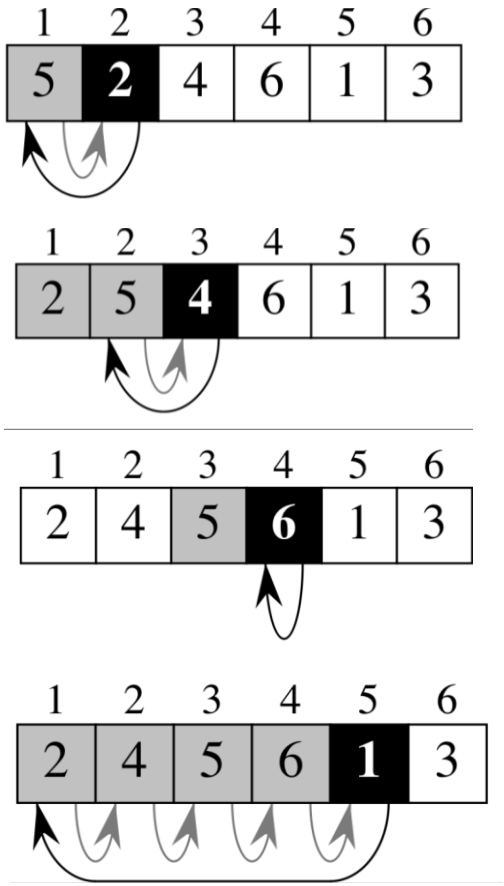

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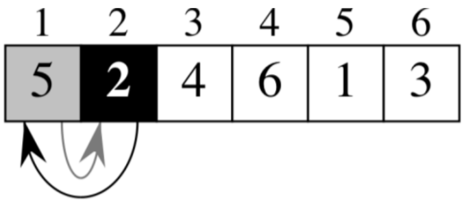

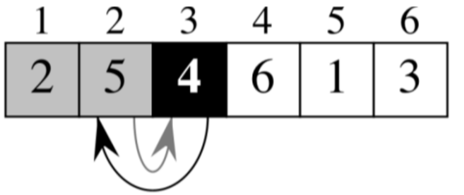

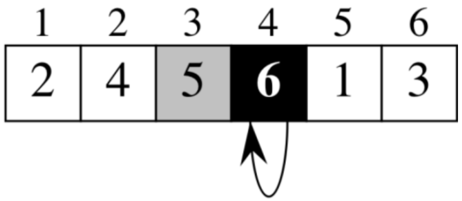

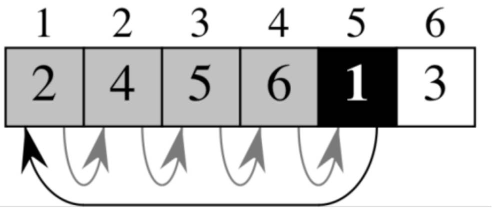

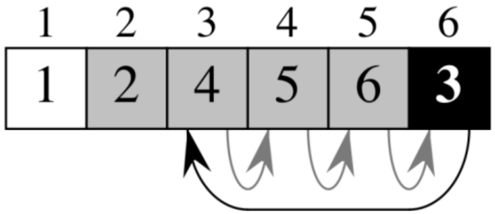

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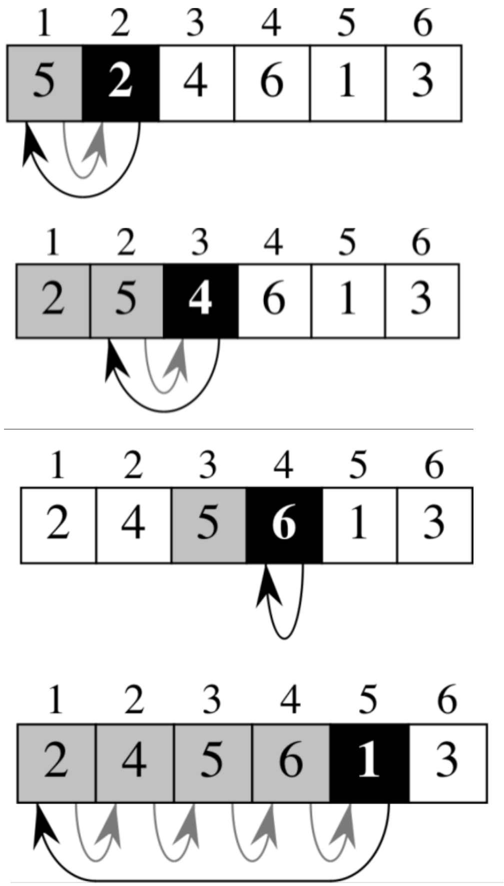

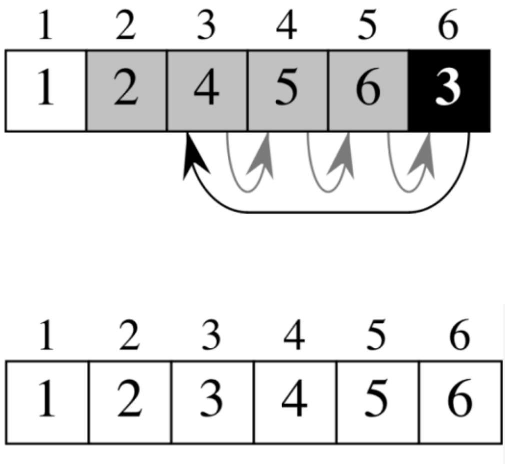

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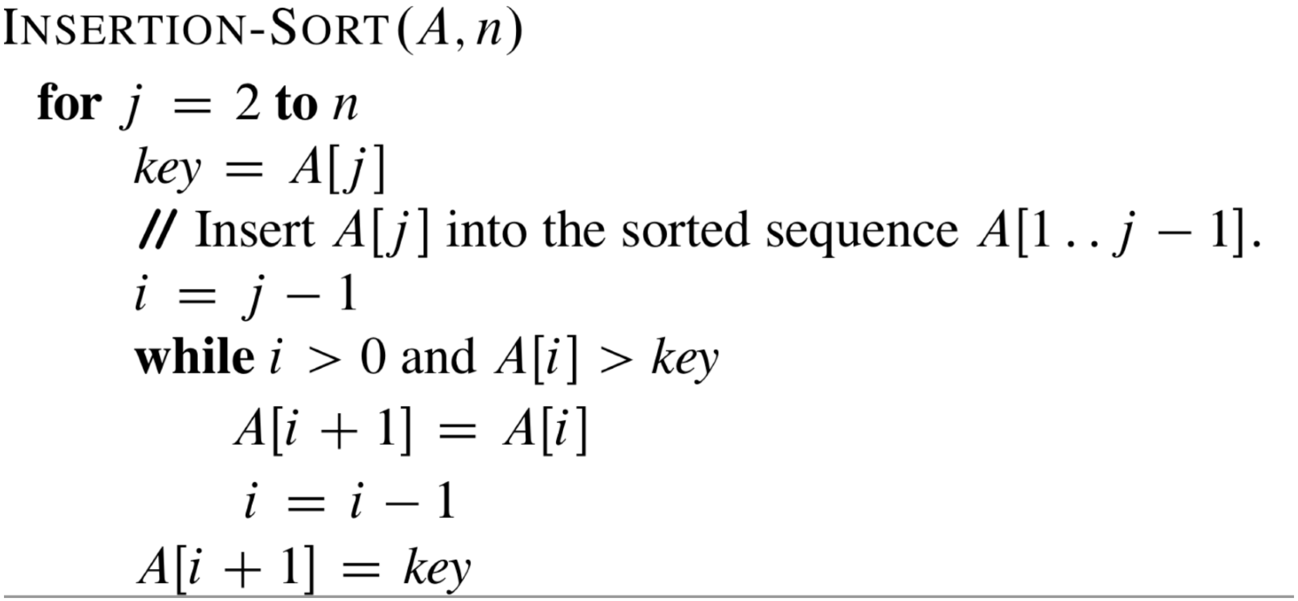
- Insertion sort loop invariant (ISLI):
  - The first (j-1) array elements A[1..j-1] are:
    (a) the original (j-1) elements, and (b) sorted.



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- Loop invariant (LI) correctness

1. Initialization:

   LI is true prior to the 1st iteration.

2. Maintenance:

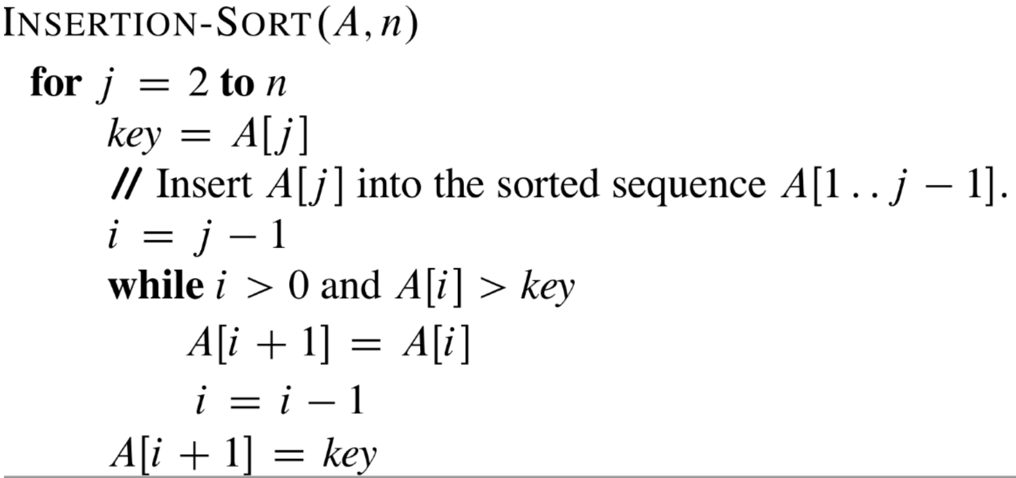
   If LI true before the iteration, it remains true before the next iteration

3. Termination:

   After the loop terminates, the output is correct.

Algorithm Correctness

- ISLI: The first (j-1) array elements A[1..j-1] are:
  (a) the original (j-1) elements, and (b) sorted.
  1. Initialization:
     Prior to the 1st iteration, j=2, the first (2-1) is sorted by definition.
  2. Maintenance:
     The (j-1)th iteration inserts the jth element in a sorted order, so after the iteration, the first (j-1) elements remains the same and sorted.
  3. Termination:
     The loop terminates after (n-1) iterations, j=n+1, so the first n elements are sorted, then the output is correct.



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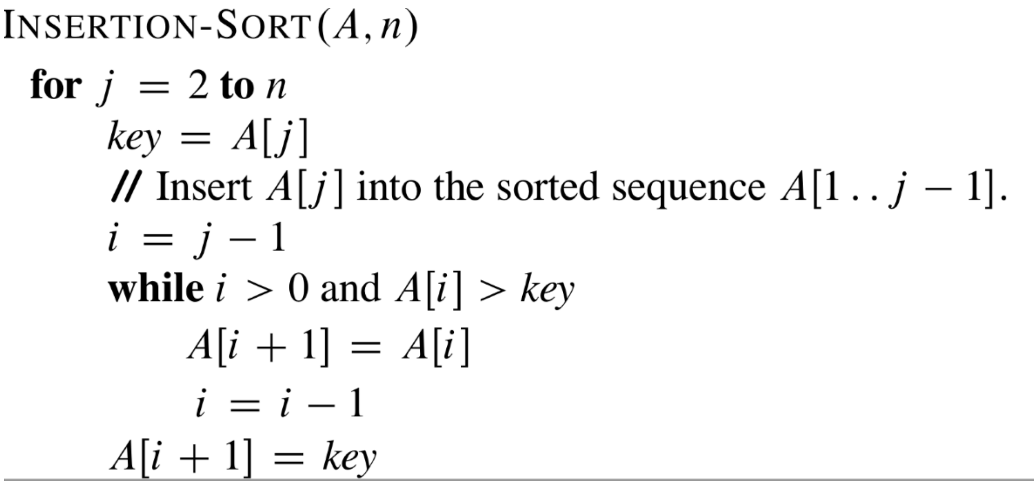
Correct



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✓

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Algorithms Performance Analysis

- Which criteria should be taken into account?

- Running time
- Memory footprint
- Disk IO
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- Lines of codes
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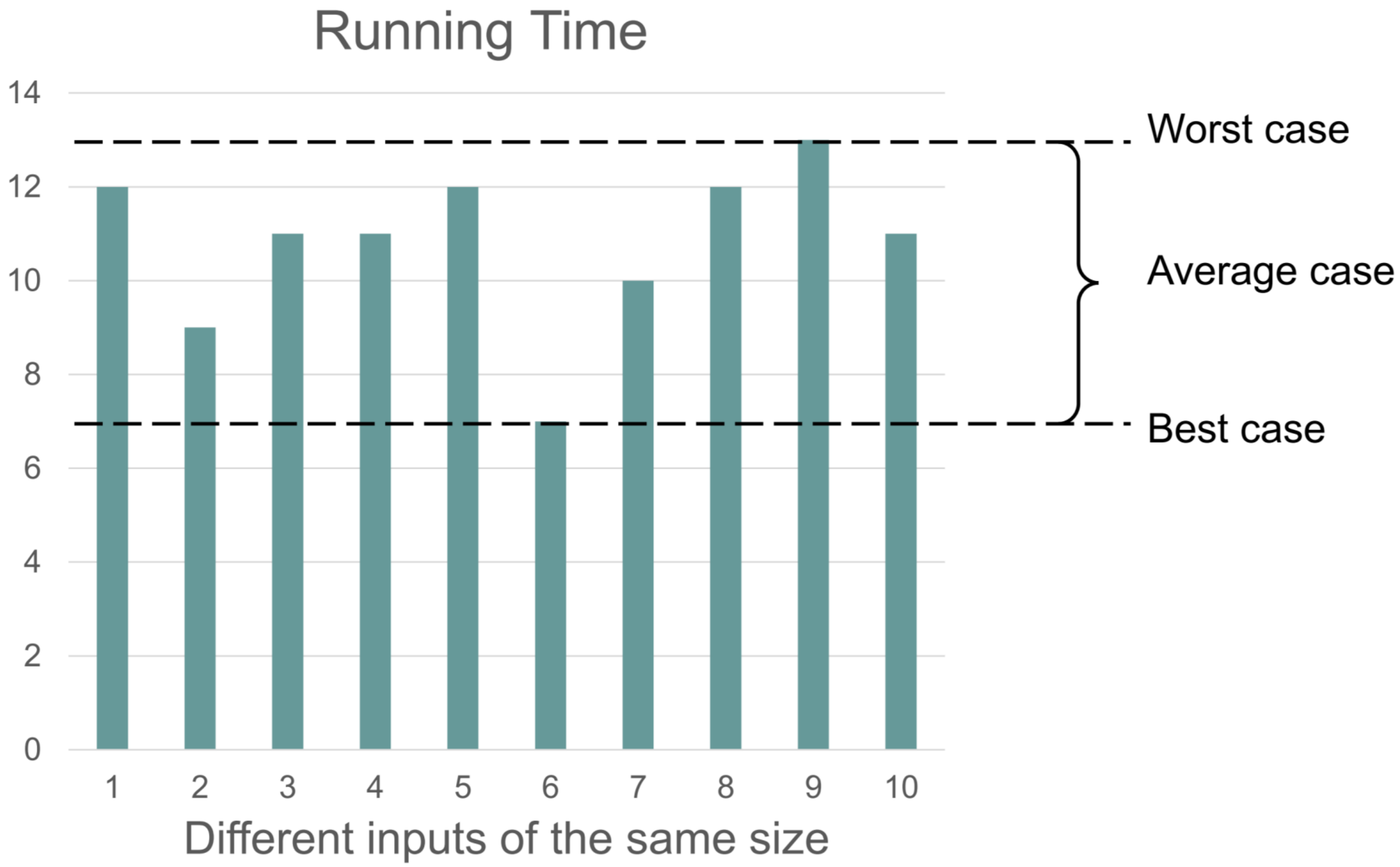
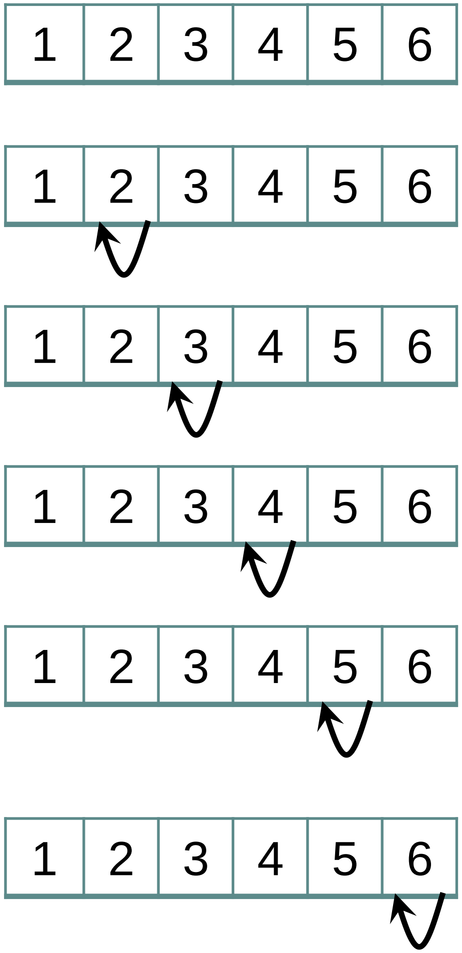
Average Case vs. Worst Case



Insertion Sort Best Case

- Input array is sorted



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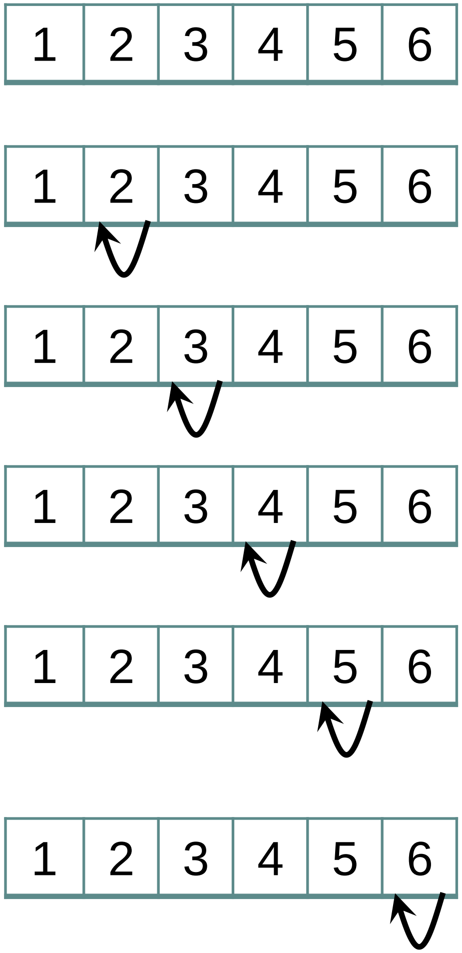

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INSERTION-SORT$(A, n)$
for $j = 2$ to $n$ ..............................................c1
    $key = A[j]$ .................................................c2
    // Insert $A[j]$ into the sorted sequence $A[1..j-1]$..........0
    $i = j - 1$ ..................................................c3      (n-1)
    while $i > 0$ and $A[i] > key$ ...........................c4
        $A[i+1] = A[i]$
        $i = i - 1$          do not execute .......... 0          1
    $A[i+1] = key$ ..........................................c5
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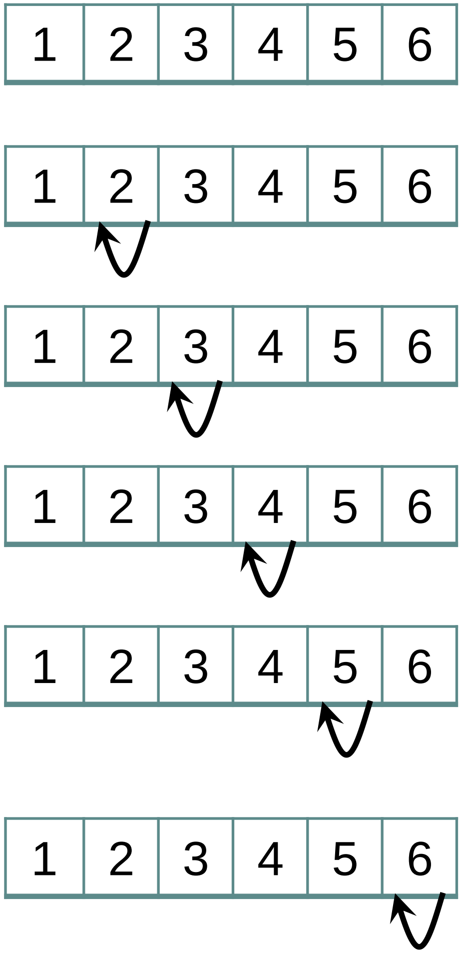

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1

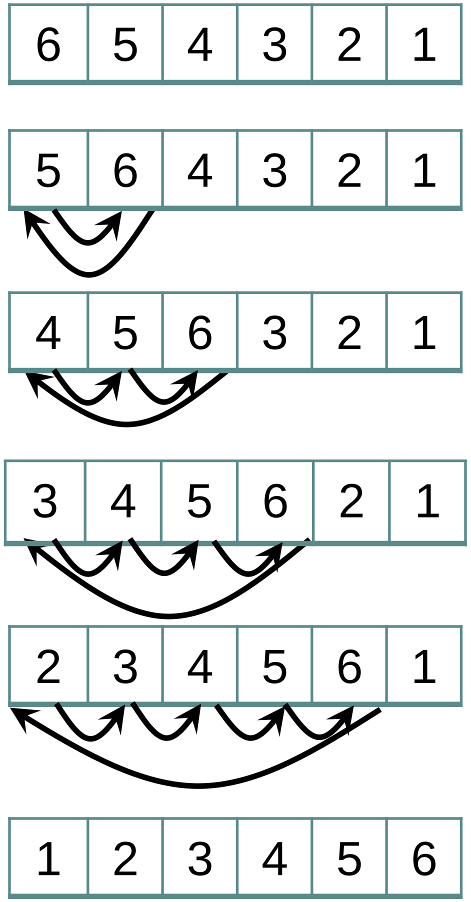
(n-1)

T(n) = (n-1)*(c1+c2+0+c3+1*(c4+0)+c5)

T(n) = cn-c, const c=c1+c2+c3+c4+c5

Insertion Sort Worst Case

- Input array is reversed



6	5	4	3	2	1
5	6	4	3	2	1
4	5	6	3	2	1
3	4	5	6	2	1
2	3	4	5	6	1
1	2	3	4	5	6

Insertion Sort Worst Case

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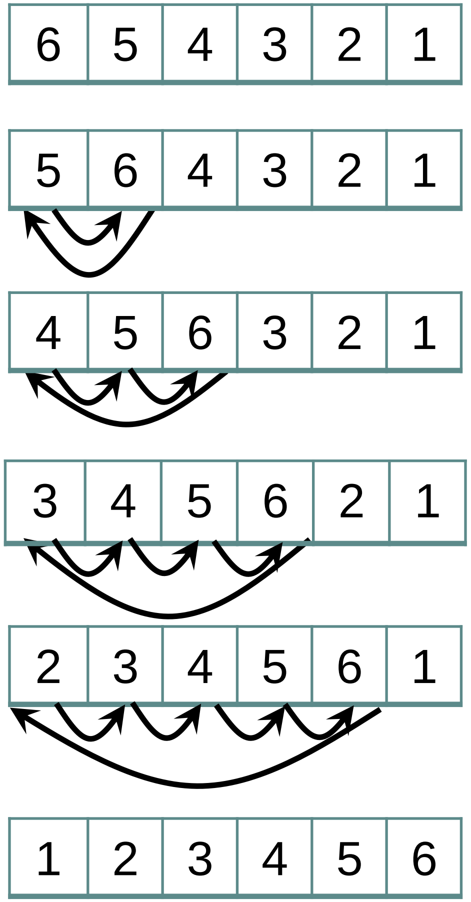

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                                                                         i
        $i = i - 1$ ..............................c6
    $A[i+1] = key$ ................................................c7
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Insertion Sort Worst Case

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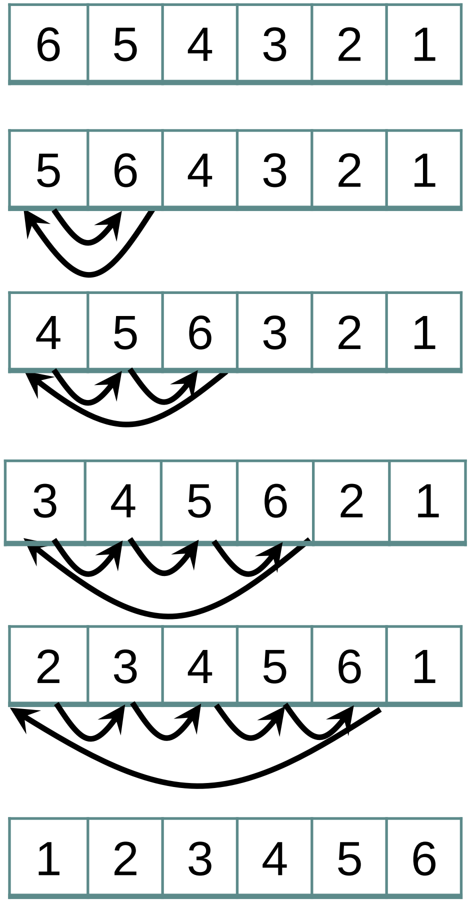

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```

(n-1)

i

$T(n) = (n-1)*(c1+c2+0+c3+i*(c4+c5+c6)+c7)$

$T(n) = (n-1)*(c1+c2+0+c3+c7) + \sum i*(c4+c5+c6)$, for all $1 <= i < n$

$T(n) = (cn-c) + \sum i*d$, c & d are constants

$\sum i*d = 1*d+2*d+3*d+....+(n-1)*d= d *(1+2+3+...(n-1))= d*n(n-1)/2$

$T(n) = (cn-c) + dn^2/2-dn/2 = d*n^2+c11*n+c12$, c's & d are consts

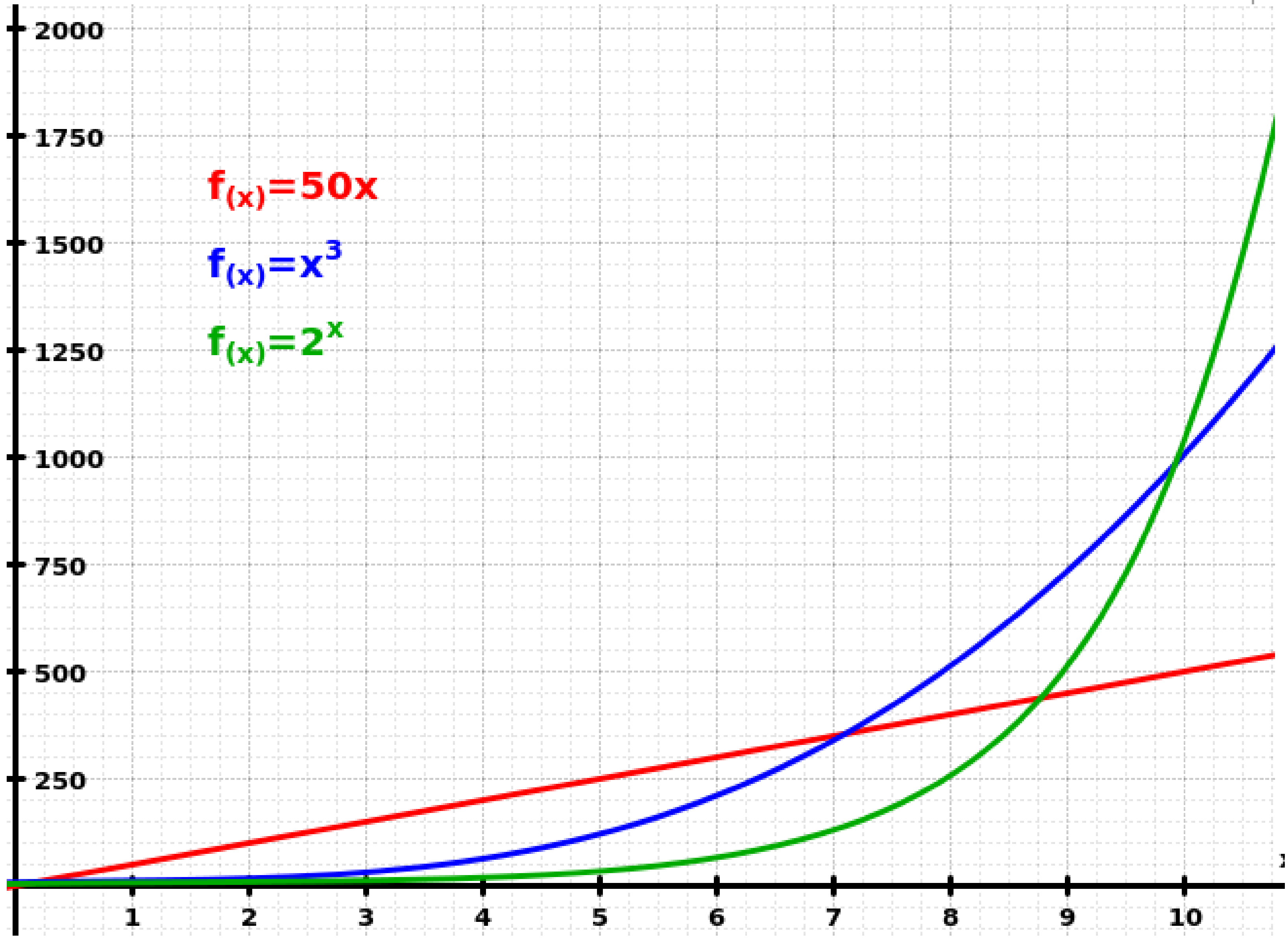
Insertion Sort Average Case

- Average = (Best + Worst)/2
- $T(n) = cn^2+dn+e$,   c, d, e are consts

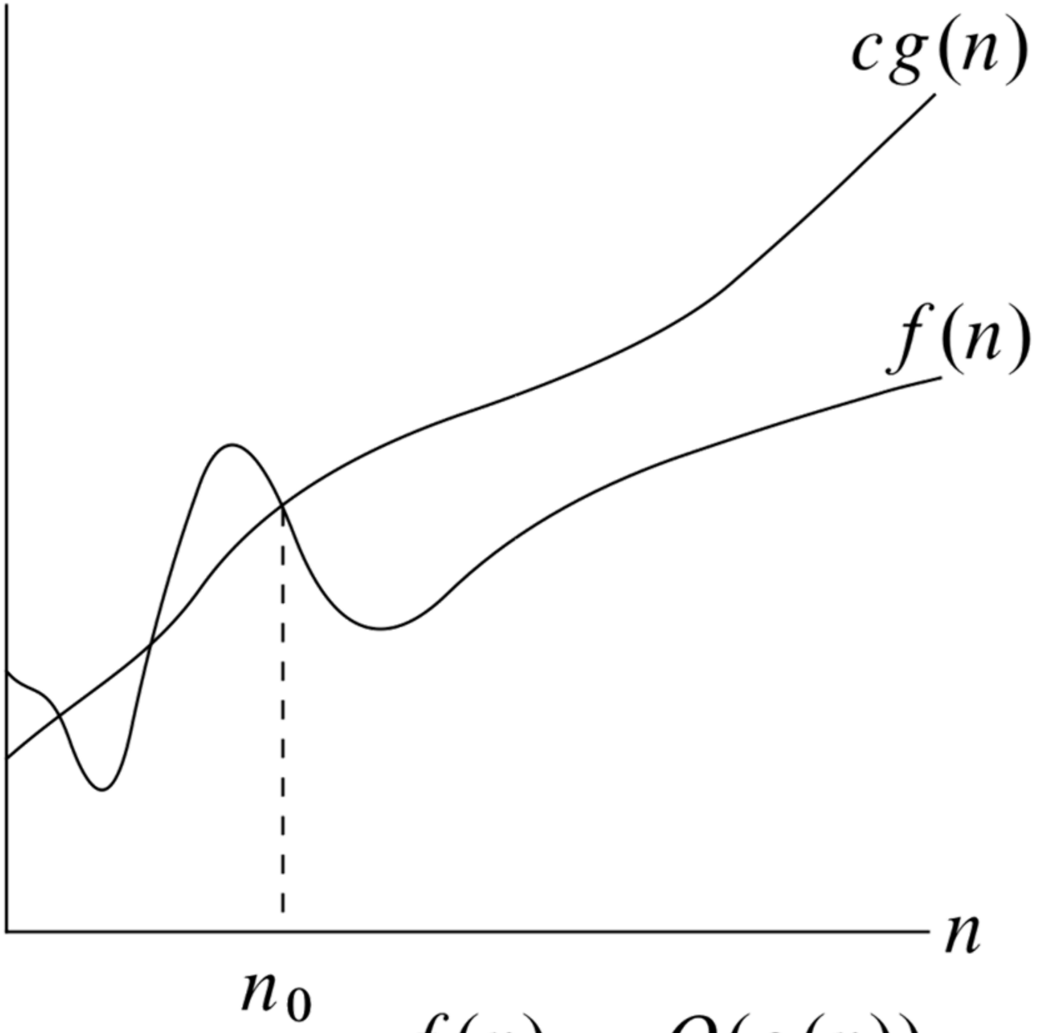
Growth of Functions

- It is hard to compute the actual running time for more complex algorithms
- The cost of the worst-case is a good measure
- The growth of the cost function is what interests us (when input size is large)
- We are more concerned with comparing two cost functions, i.e., two algorithms.

Growth of Functions



O-notation

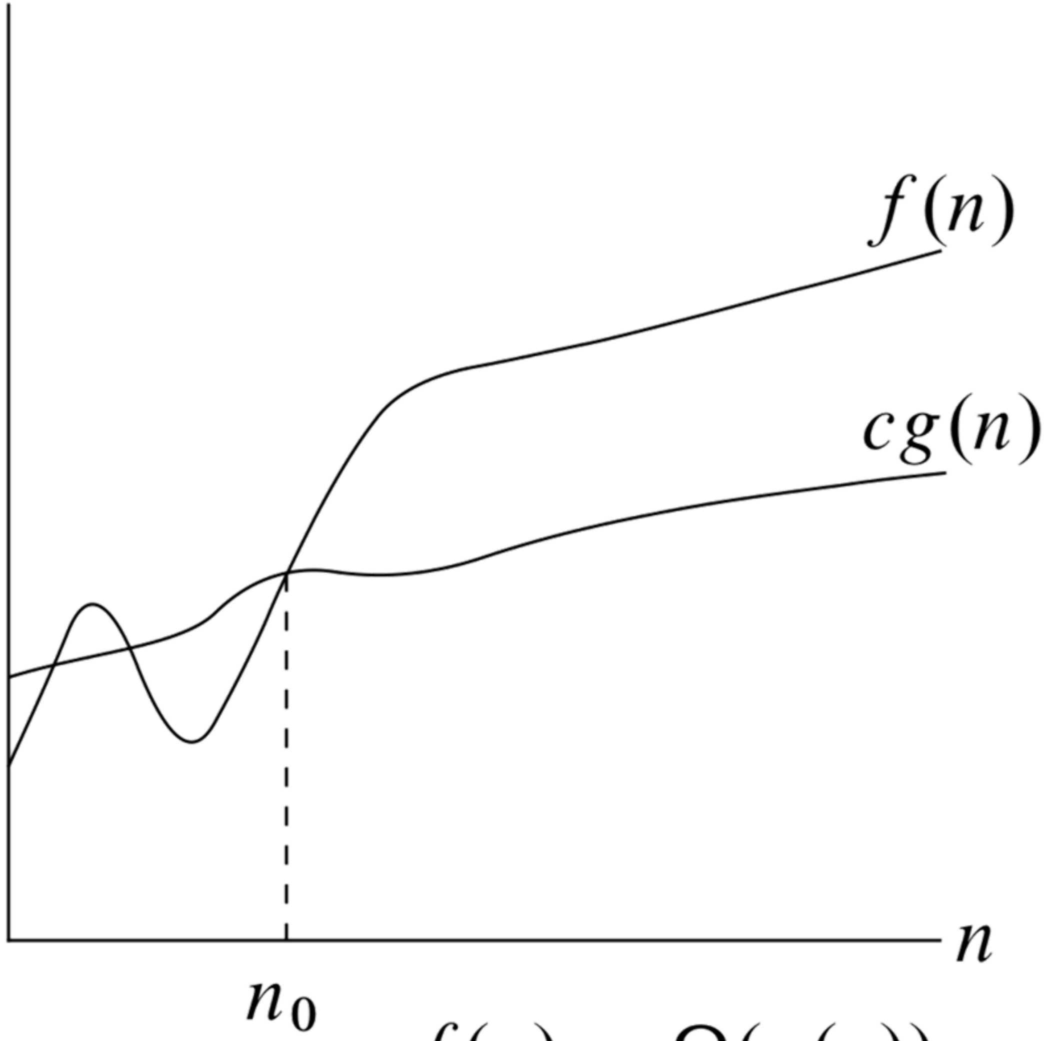


$f(n) = O(g(n))$

$$\exists c > 0, n_0 > 0$$
$$0 \leq f(n) \leq cg(n)$$
$$n \geq n_0$$

g(n) is an asymptotic upper-bound for f(n)

Ω-notation



$f(n) = \Omega(g(n))$

$\exists c > 0, n_0 > 0$
$0 \le cg(n) \le f(n)$
$n \ge n_0$

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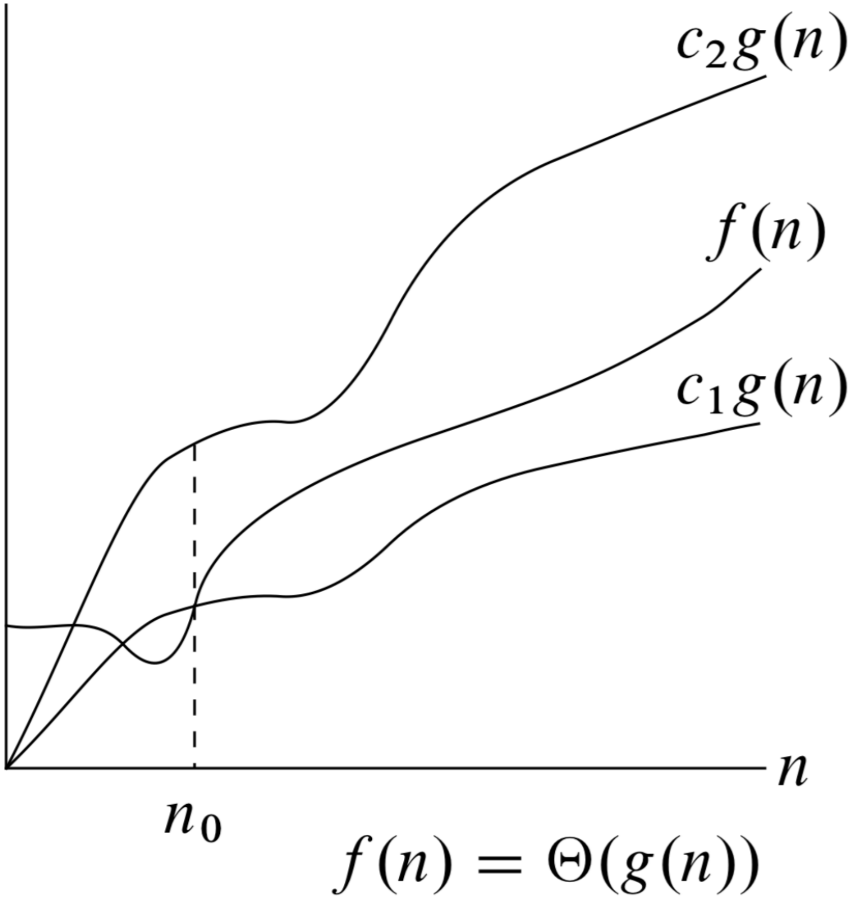
Θ-notation

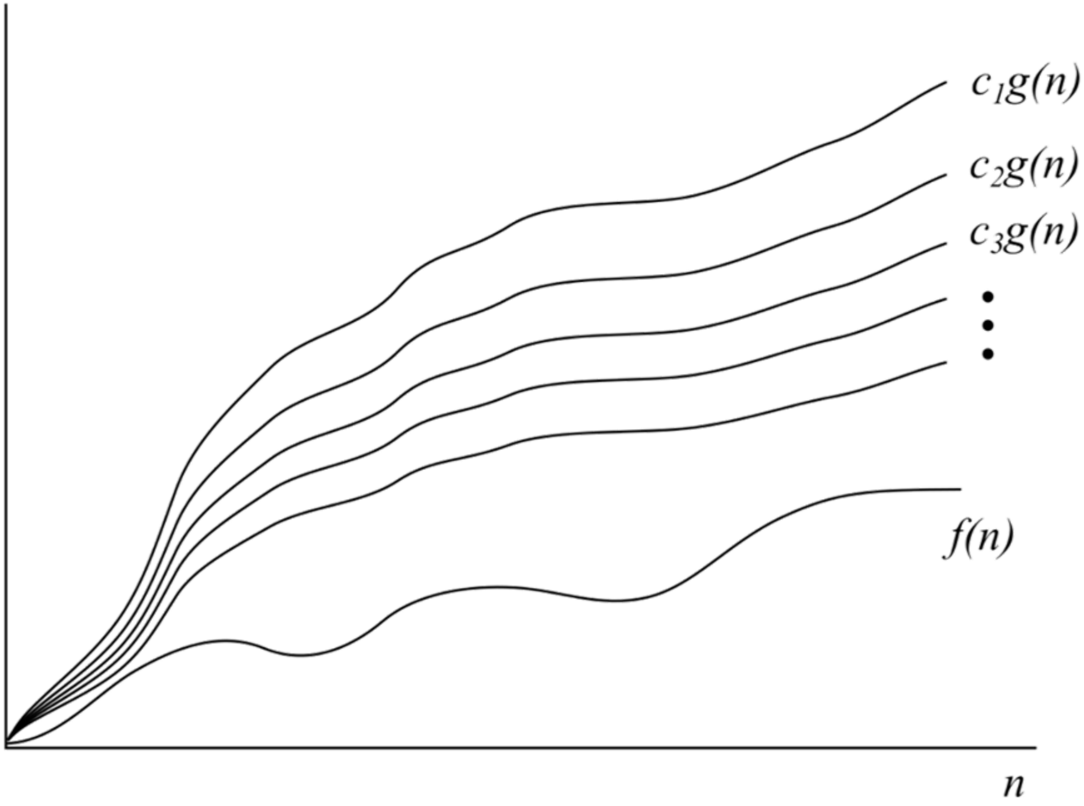


$$\exists c_1, c_2 > 0, n_0 > 0$$
$$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$$
$$n \ge n_0$$

g(n) is an asymptotic tight-bound for f(n)

o-notation

$$\forall c > 0$$
$$\exists n_0 > 0$$
$$0 \le f(n) \le cg(n)$$
$$n \ge n_0$$



$$f(n) = o(g(n))$$

g(n) is a non-tight asymptotic upper-bound for f(n)

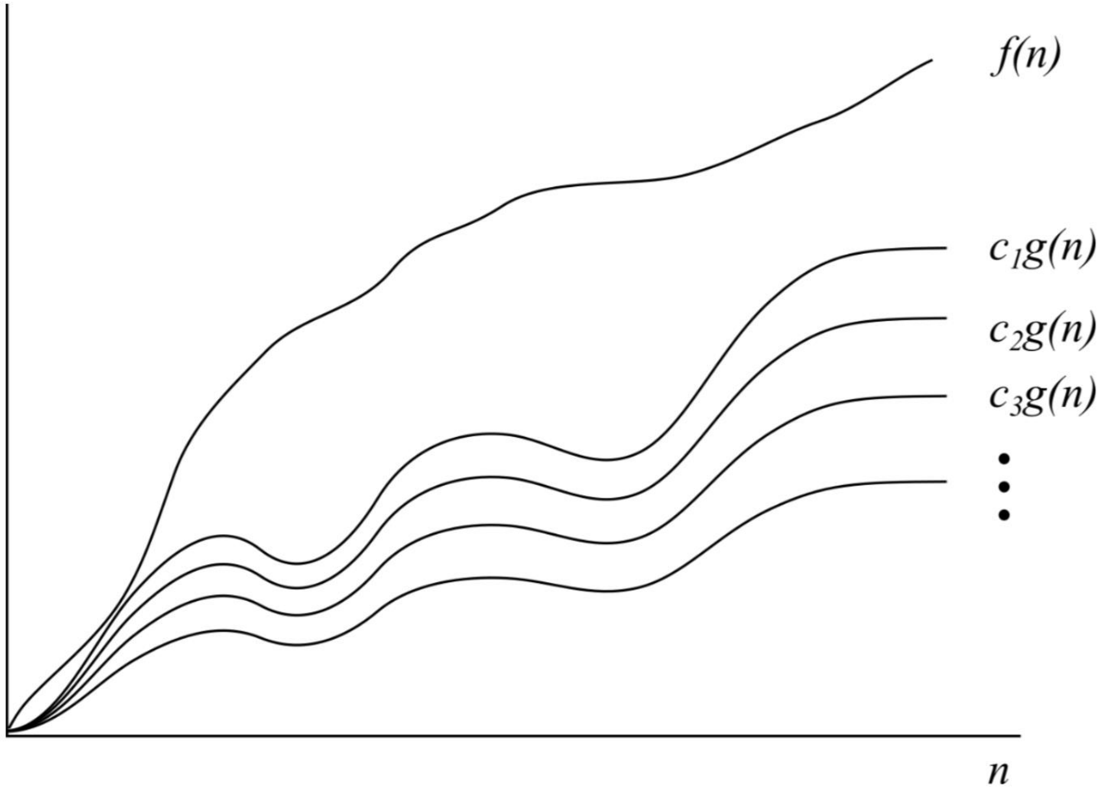
ω-notation

$$\boxed{\forall c > 0}$$

$$\exists n_0 > 0$$

$$0 \le cgn(n) \le f(n)$$

$$n \ge n_0$$



$$f(n) = \omega(g(n))$$

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Comparing Two Functions

- $lim_{n\to\infty} \frac{f(n)}{g(n)}$
- 0: f(n) = o(g(n))
- c > 0: f(n) = Θ(g(n))
- ∞: f(n) = ω(g(n))

Analogy to Real Numbers

Functions	Real numbers
$\boldsymbol{f(n) = O(g(n))}$	$a \leq b$
$\boldsymbol{f(n)} = \Omega(\boldsymbol{g(n)})$	$a \geq b$
$\boldsymbol{f(n)} = \Theta(\boldsymbol{g(n)})$	$a = b$
$\boldsymbol{f(n)} = \mathrm{o}(\boldsymbol{g(n)})$	$a < b$
$\boldsymbol{f(n)} = \omega(\boldsymbol{g(n)})$	$a > b$

Simple Rules

- We can omit constants
- We can omit lower order terms
- $\Theta(an^2+bn+c)$ becomes $\Theta(n^2)$
- $\Theta(c1)$ and $\Theta(c2)$ become $\Theta(1)$
- $\Theta(\log_{k1} n)$ and $\Theta(\log_{k2} n)$ become $\Theta(\log n)$
- $\Theta(\log(n^k))$ becomes $\Theta(\log n)$
- $\log^{k1}(n) = o(n^{k2})$ for any positive constants $k1$ and $k2$

Popular Classes of Functions

Constant: $f(n) = \Theta(1)$

Logarithmic: $f(n) = \Theta(\lg(n))$

Sublinear: $f(n) = o(n)$

Linear: $f(n) = \Theta(n)$

Super-linear: $f(n) = \omega(n)$

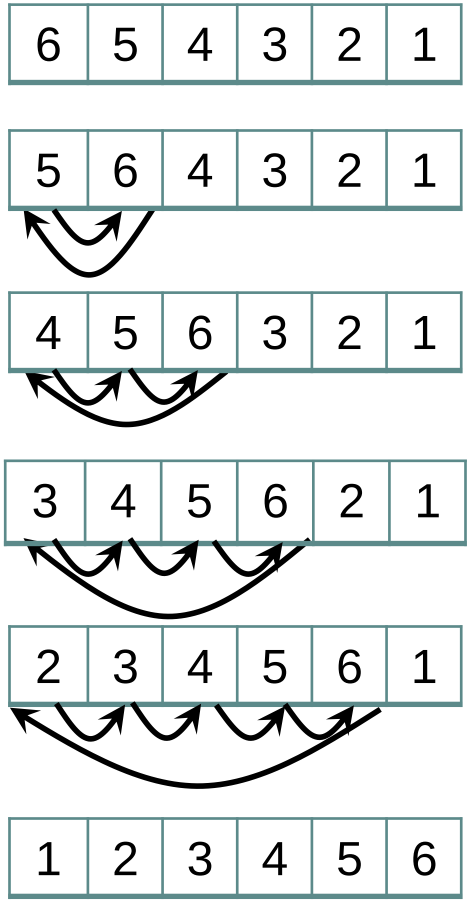
Quadratic: $f(n) = \Theta(n^2)$

Polynomial: $f(n) = \Theta(n^k)$; k is a constant

Exponential: $f(n) = \Theta(k^n)$; k is a constant

Insertion Sort Worst Case (Revisit)

- Input array is reversed



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max n

(n-1)

T(n) = (n-1)*n = O(n^2)

Comparing two algorithms

- T1(n) = 2n+1000000
- T2(n) = 200n + 1000
- Which is better? Why?
  - In terms of order of growth?
  - In terms of actual runtime?
- What is the main usage of asymptotic notation analysis?

Analyzing Algorithms

- Algorithm 1

```
for i = 1 to n
    j = 2*i
for j = 1 to n/2
    print j
```

Analyzing Algorithms

- Algorithm 2

```
for i = 1 to n/2
    for j = 1 to n, step j = j*2
        print i*j
```

Analyzing Algorithms

- Algorithm 3

```
input x (+ve integer)
while x > 0
    print x
    $x = \lfloor x/5 \rfloor$
```

Credits & Book Readings

- Book Readings
  - 2.1, 2.2, 3.1, 3.2
- Credits
  - Prof. Ahmed Eldawy notes
    - http://www.cs.ucr.edu/~eldawy/17WCS141/slides/CS141-1-09-17.pdf
  - Online websites
    - https://commons.wikimedia.org/wiki/File:Exponential.svg