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CS 141: Intermediate Data Structures and Algorithms

Discussion - Week 4, Winter 2018

Master Theorem

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- Master Theorem
- Examples
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Motivation

- We only care about asymptotic behavior when analyzing algorithms.
- Rather than analyzing exactly the recurrence relation, we only need to find asymptotic behavior.
- Master Theorem is a great tool to do that!

Master theorem

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n),$$

where $a > 1$, $b > 1$, and f is asymptotically positive.

Master theorem

$$T(n)=\begin{cases} \Theta\left(n^{\log_b a}\right) & f(n)=O\left(n^{\log_b a-\varepsilon}\right) \\ \Theta\left(n^{\log_b a}\log n\right) & f(n)=\Theta\left(n^{\log_b a}\right) \\ \Theta\left(f(n)\right) & f(n)=\Omega\left(n^{\log_b a+\varepsilon}\right) \text{ AND} \\ & af(n/b)<cf(n) \text{ for large } n \end{cases} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \begin{matrix} \varepsilon>0 \\ c<1 \end{matrix}$$

Example 1

$T(n) = 4T(n/2) + n$

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$a = 4$, $b = 2 \Rightarrow n^{logb(a)} = n^2$; $f(n) = n$.

CASE 1: $f\,(n) = O(n^{2 - \varepsilon})$ for $\varepsilon = 1$.

$\rightarrow T(n) = (n^2)$.

Example 2

$$T(n) = 4T(n/2) + n^2$$

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$a = 4$, $b = 2 \Rightarrow n^{logb(a)} = n^2$; $f(n) = n^2$.

CASE 2: $f(n) = \Theta(n^2)$.

$\rightarrow T(n) = (n^2 \log n)$.

Example 3

$T(n) = 4T(n/2) + n^3$

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$T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \Rightarrow n^{logb(a)} = n^2; f(n) = n^3.$

CASE 3: $f (n) = \Omega(n^{2 + \varepsilon})$ for $\varepsilon = 1$.

and $4(n/2)^3 \leq cn^3$ for $c = 1/2$.

$\rightarrow T(n) = (n^3).$

Example 4

$$T(n) = 4T(n/2) + n^2 / \log n$$

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$T(n) = 4T(n/2) + n^2 / \log n$

$a = 4$, $b = 2 \Rightarrow n^{\log_b(a)} = n^2$; $f(n) = n^2 / \log n$.

Master theorem does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\log n)$.

Group activities

Solve and discuss several recurrence relations.