CS141: Intermediate Data Structures and Algorithms

NP-Completeness

Amr Magdy

Why Studying NP-Completeness?

- Two reasons:
  1. In almost all cases, if we can show a problem to be **NP-complete** or **NP-hard**, the best we can achieve (NOW) is mostly exponential algorithms.
     - This means we cannot solve large problem sizes efficiently
  2. If we can solve only one NP-complete problem efficiently, we can solve ALL NP problems efficiently (major breakthrough)
- More details come on what does these mean

Topic Outline

1. Background
   - Decision vs. Optimization Problems
   - Models of Computation
   - Input Encoding
2. Complexity Classes
   - P
   - NP
     - Polynomial Verification
     - Examples
3. NP-hardness
   - Polynomial Reductions
4. NP-Complete Problems
   - Definition and Examples
   - Weak vs. Strong NP-Complete Problems

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- Converting optimization problem → decision problem?
  - Put a bound on the objective function.
    - Does G have a clique of size k? for k= 3, 4, 5,…(finding max clique)

Take Home Messages

(1) Computation theory focuses on decision problems

Models of Computation

- MoC: informally a theoretic description of a way to compute

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- Example: mask model

Mask Model (on paper)

Mask Realization (fabric instance)

Models of Computation

- At a low level:
  - Finite State Automata (FSA)
  - Pushdown Automata (PDA)
  - Turing Machine (TM)
  - …..

Focus of other courses (e.g., Theory of Computation, Compilers Design, ...etc)

- At a high level:
  - RAM (Random Access Machine)
  - Pointer Machine
  - ….

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  - Possible operations in $\Theta(1)$:
    - Access any memory word at random
    - Read variable
    - Write variable
    - Basic mathematical operations (add, multiply, assign,…etc)
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  - Function of the basic operations

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(2) Algorithm complexity is affected by the computation model

Input / Output Encoding

- Assume multiplying two decimal integers
  - 2 * 2 = 4

    (basic operation, single digit op)
  - 12*12 = (1*10+2)*(1*10+2)

    = 1*10*1*10+1*10*2+2*1*10+2*2

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  - $O(n^2)$ operations for n-digit number
- Input representation (encoding) affects the amount of computations for same input

Same input (2x2), different encoding

Exercise

- design a divide & conquer algorithm to multiply two n-bits integers in $O(n^2)$

- Note:
  - Multiplying by 2^n for binary numbers is shifting by n bits → $\Theta(n)$
  - Multiplying by 10^n for decimal numbers is shifting by n digits → $\Theta(n)$

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Encoding Examples in Binary Strings

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- Graph G of n vertices and m edges:
  - Each vertex with integer id → n integers
  - Each edge with integer id and weight → m integers + m floats
  - m is maximum of $n^2/2$, i.e., $m=O(n^2)$
  - Example: 0110101000001110110111001000011101011110010000 1110011010100000111011011110010000111010111111001001…

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Concrete input string

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(3) Binary input string (concrete input) is different in length than the algorithm abstract input

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  - Either in time complexity
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- In this course, our discussion is limited to only two time complexity classes: P and NP
  - Other courses cover more content (e.g., Theory of Computation course)

P

- P is a complexity class of problems that are ***decidable*** in **polynomial-time** of the concrete input string length, i.e., $O(b^k)$
  - where <u>b the binary (concrete) input string length</u> and <u>k is constant</u>

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- Examples: (translate the complexity in terms of concrete input length not abstract input length)
  - Shortest paths in graph
  - Matrix chain multiplication
  - Activity scheduling problem
  - ….

P

- As long as the algorithm complexity is polynomial in terms of the concrete input length, it belongs to class P

- Example: Matrix chain multiplication

Abstract input length a= (n+1) integers

Concrete input length b= ~(n log n) bits

Algorithm complexity: $O(n^3) = O(a^3) = O(b^3)$

As $a^3 = \sim n^3$

$b^3 = n^3 \log^3 n$

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- The problem must be decision (not optimization) problem

- Examples:
  - Is bipartite graph? Given two subsets of nodes, verify it is bipartite
  - Max clique: Given a clique and k, verify it is actually a clique of size k
  - Shortest path: Given a path of cost C, verify it is a path and of cost C
  - …..

Is P ⊂ NP?

Is P $\subset$ NP?

- Yes
- What does this mean?

Is P ⊂ NP?

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- What does this mean?
  - Every problem that is solvable in polynomial time is verifiable in polynomial time as well

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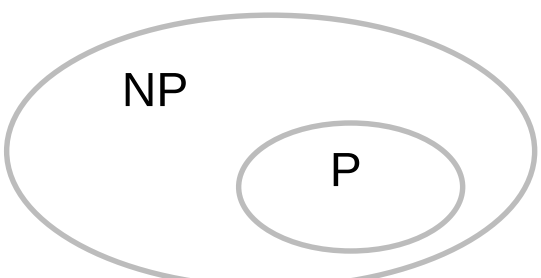
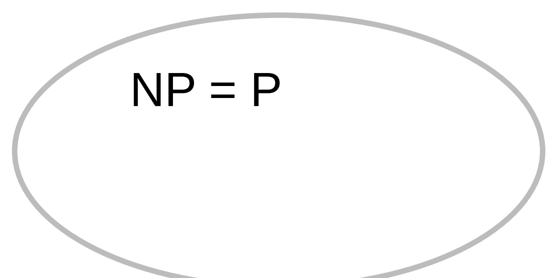
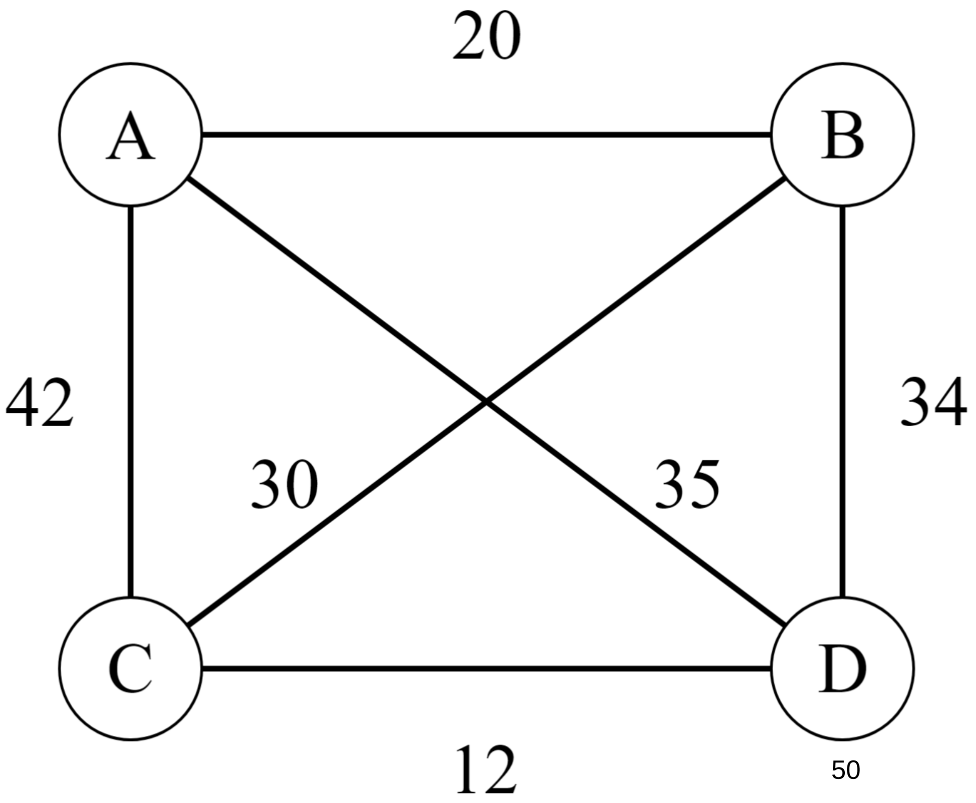




NP Problems

- Example: **Travelling Salesman Problem**

  Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?



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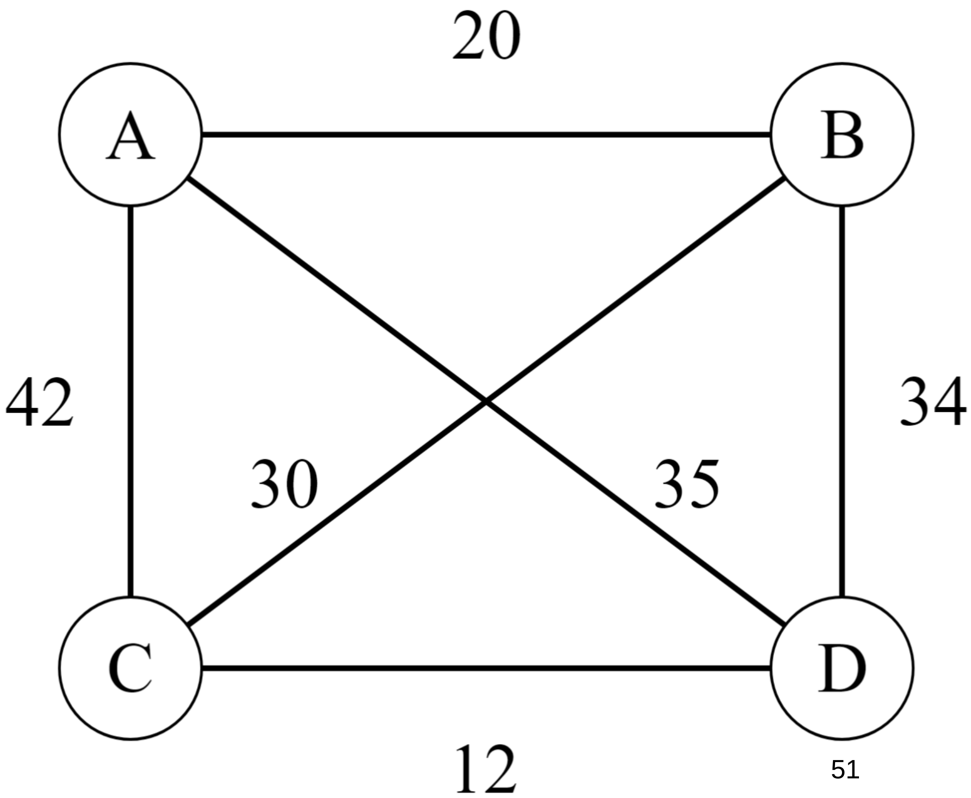

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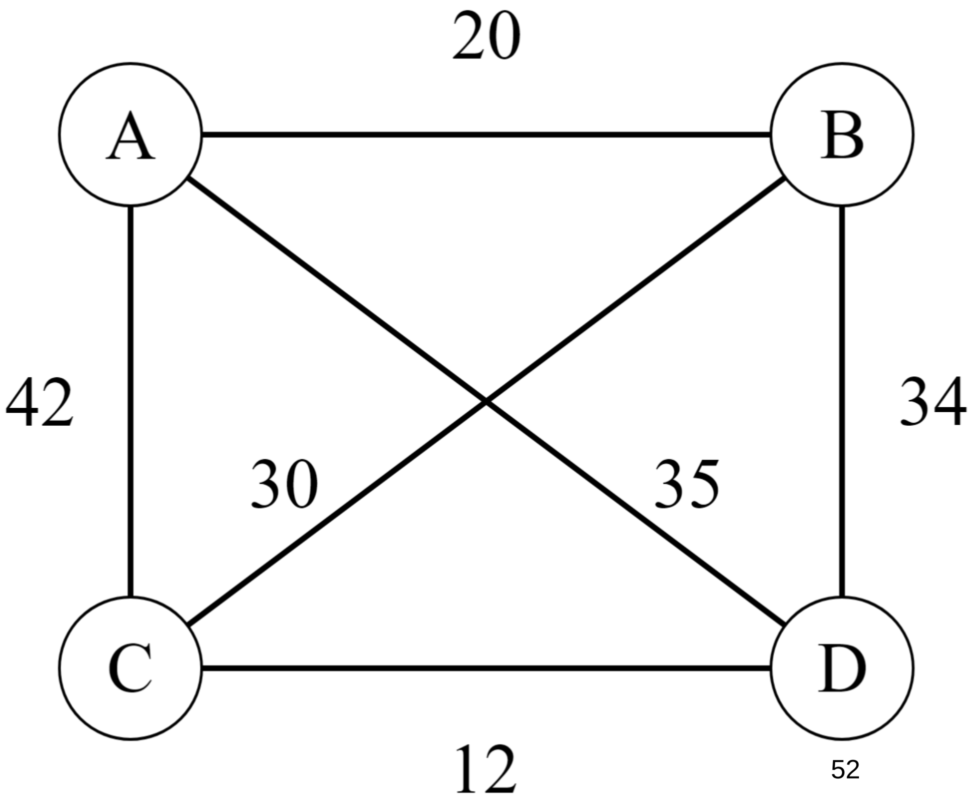

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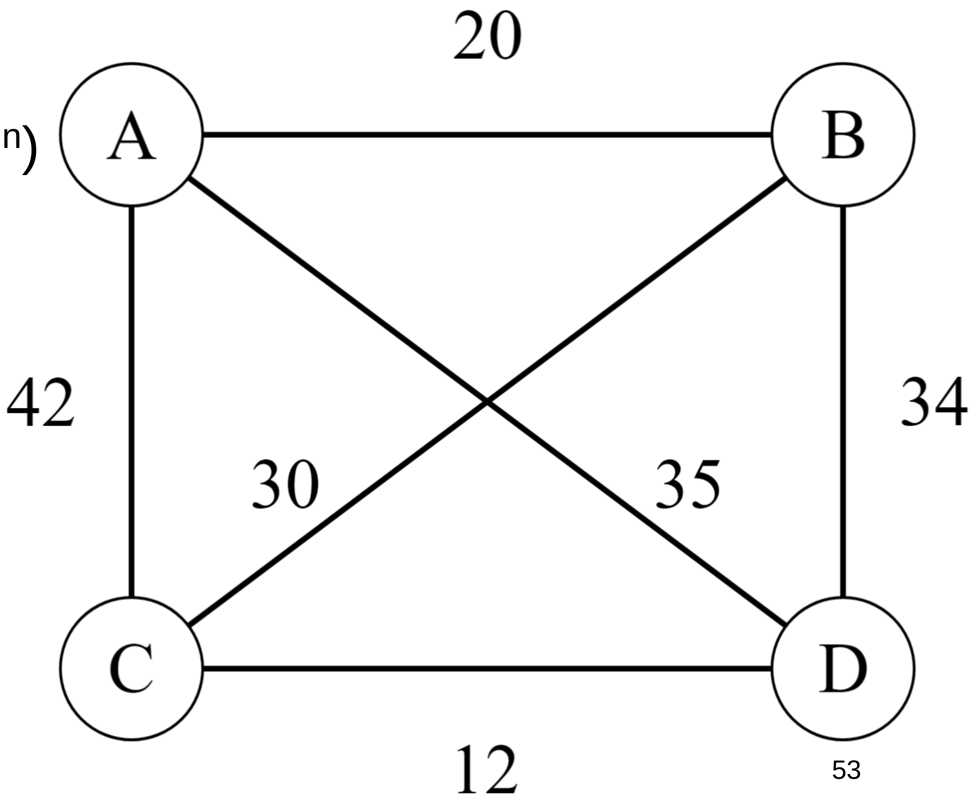
  - Dynamic programming: $O(n2^n)$



Travelling Salesman Movie

- https://www.youtube.com/watch?v=6ybd5rbQ5rU

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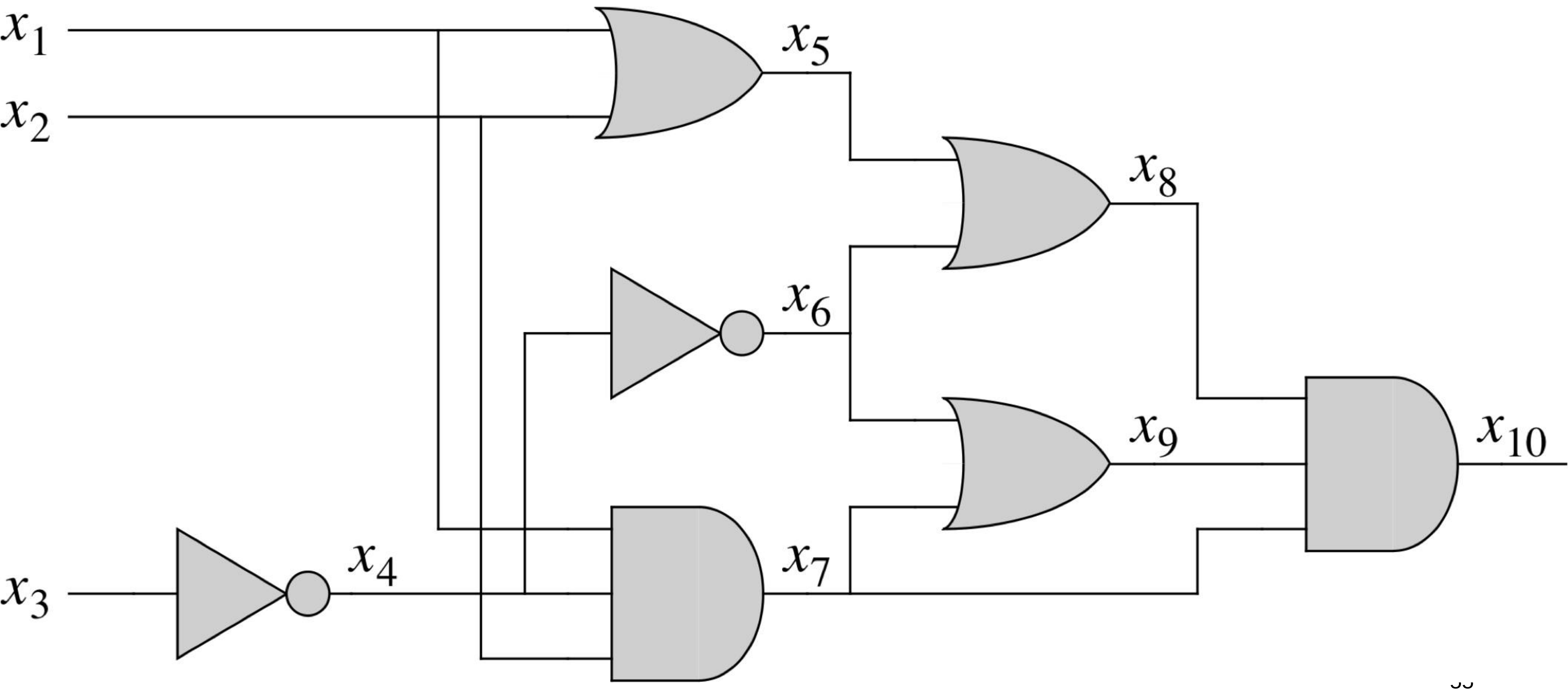
- Example: **SAT Problem**

Given a Boolean circuit S, is there a satisfying assignment for S? (i.e., variable assignment that outputs 1)



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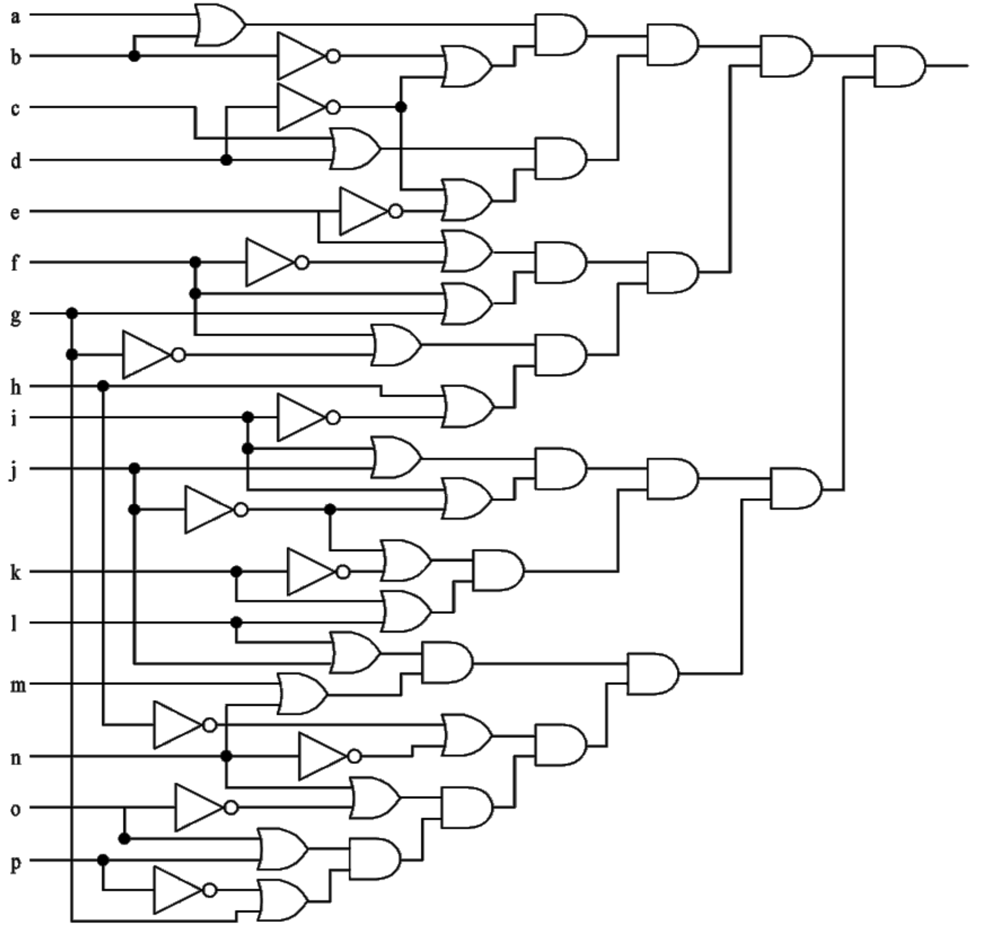

NP Problems

- Example: **3-CNF Problem**

Given a Boolean circuit S in 3-CNF form, is there a satisfying assignment for S? (i.e., variable assignment that outputs 1)

- 3-CNF formula: a set ANDed Boolean clauses, each with 3 ORed literals (Boolean variables)

- Example: $\vee$ = OR, $\wedge$ = AND, $\neg$ = NOT
  $(x1 \vee \neg x2 \vee \neg x3) \wedge (\neg x1 \vee x2 \vee x3) \wedge (x1 \vee x2 \vee x3)$

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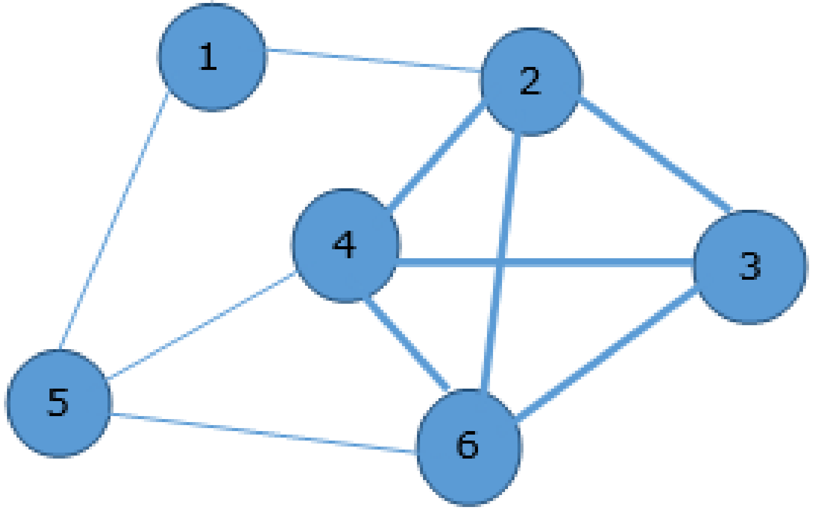
- Solution: $O(k2^n)$ for k clauses and n variables

NP Problems

- Example: **(Max) Clique Problem**

Given a graph G=(V,E), find the clique of maximum size.

Clique: fully connected subgraph.



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- Example: **(Max) Clique Problem**

Given a graph G=(V,E) of n vertices, find the clique of maximum size.

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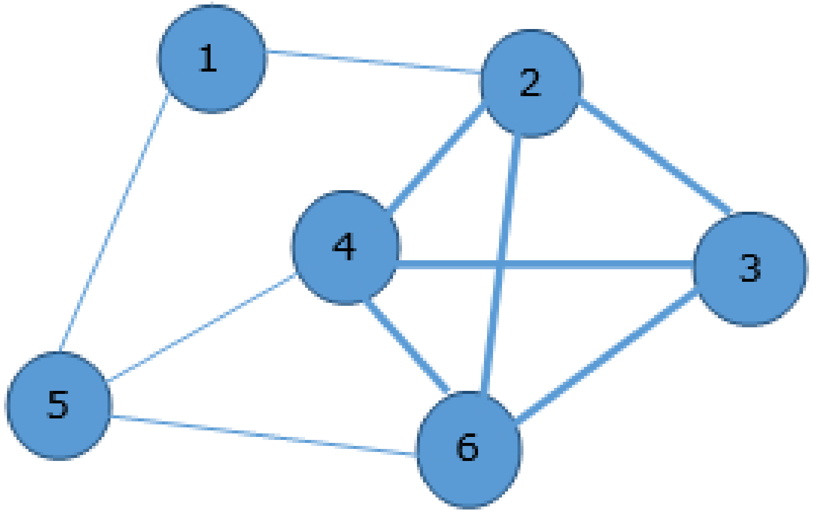

- Solution:
  - Assume max clique size k and $|V| = n$
  - Brute force: $O(n2^n)$
  - Combinations of k: $O(n^k k^2)$
    - Try for k=3,4,5,…
    - k is not constant, so this is not polynomial

NP Problems: Polynomial Verification

- Given a solution, can I verify if it is correct in polynomial time?
- TSP Problem: Yes (the decision version)
  - Is there a tour with weight W?
- SAT Problem: Yes
- 3-CNF Problem: Yes
- Max Clique Problem: Yes (the decision version)
  - Is there a clique of size k?

NP-hard Problems

- Informally:

an NP-hard problem B is a problem that is at least as hard as the hardest problems in NP class

- Formally:

B is NP-hard if $\forall$ A $\in$ NP, A $\leq_P$ B
(i.e., A is polynomially reducible to B)

Polynomial Reductions

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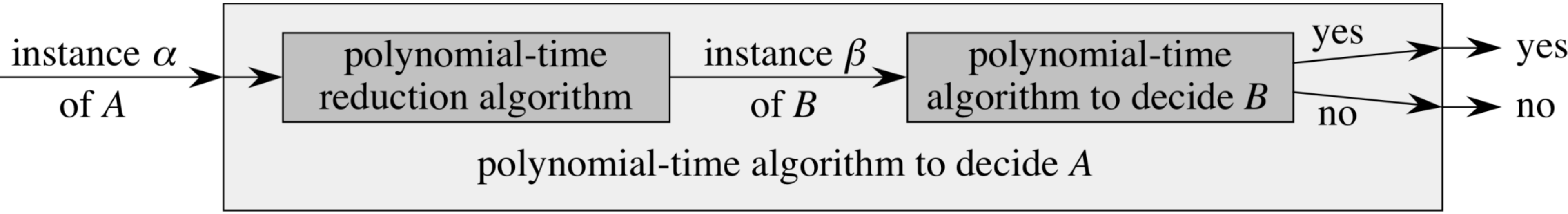

Polynomial Reductions: Example

- Reduce k-clause 3-CNF problem to k-size Clique problem
- Example: three 3-CNF clauses
  $(x1 \lor \neg x2 \lor \neg x3) \land (\neg x1 \lor x2 \lor x3) \land (x1 \lor x2 \lor x3)$

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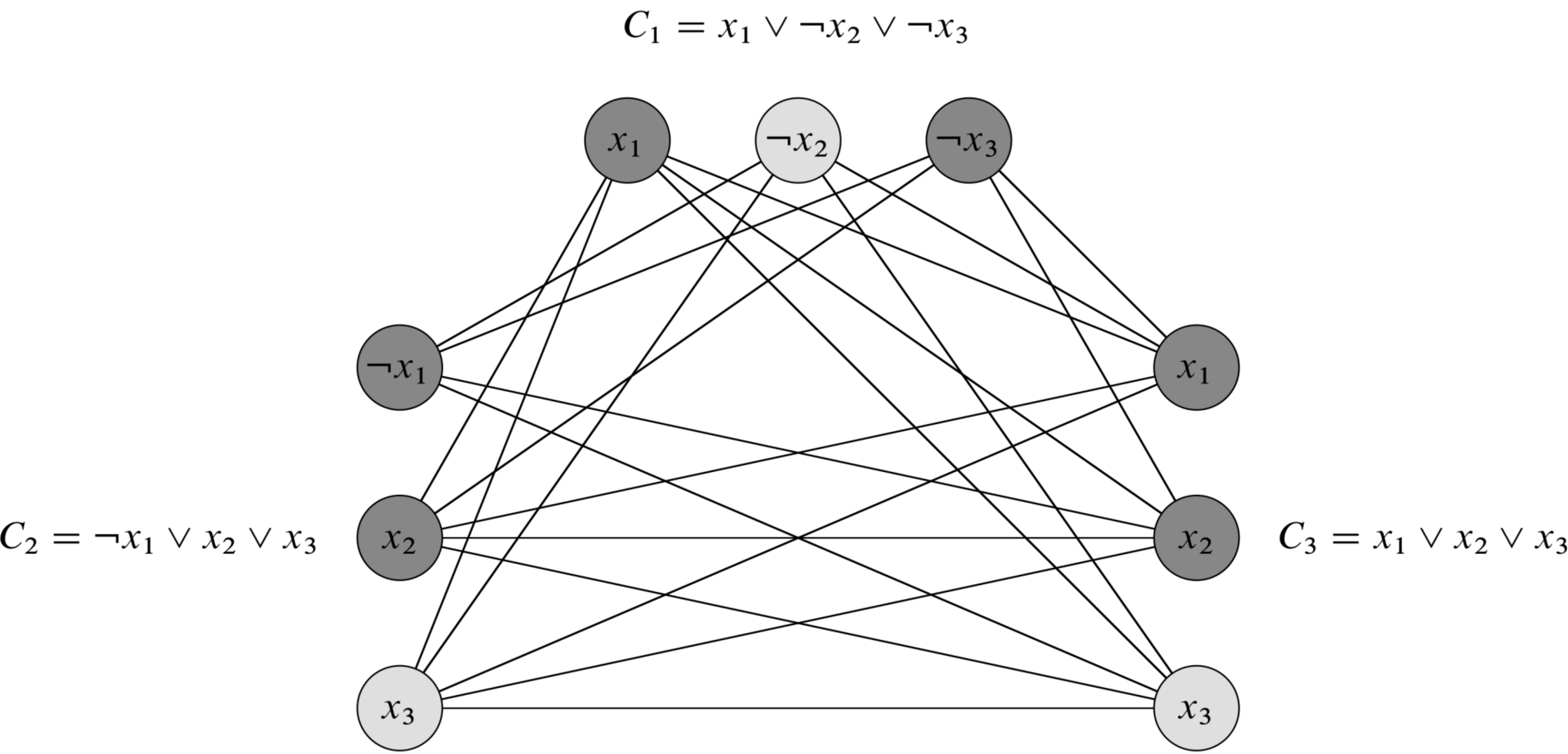

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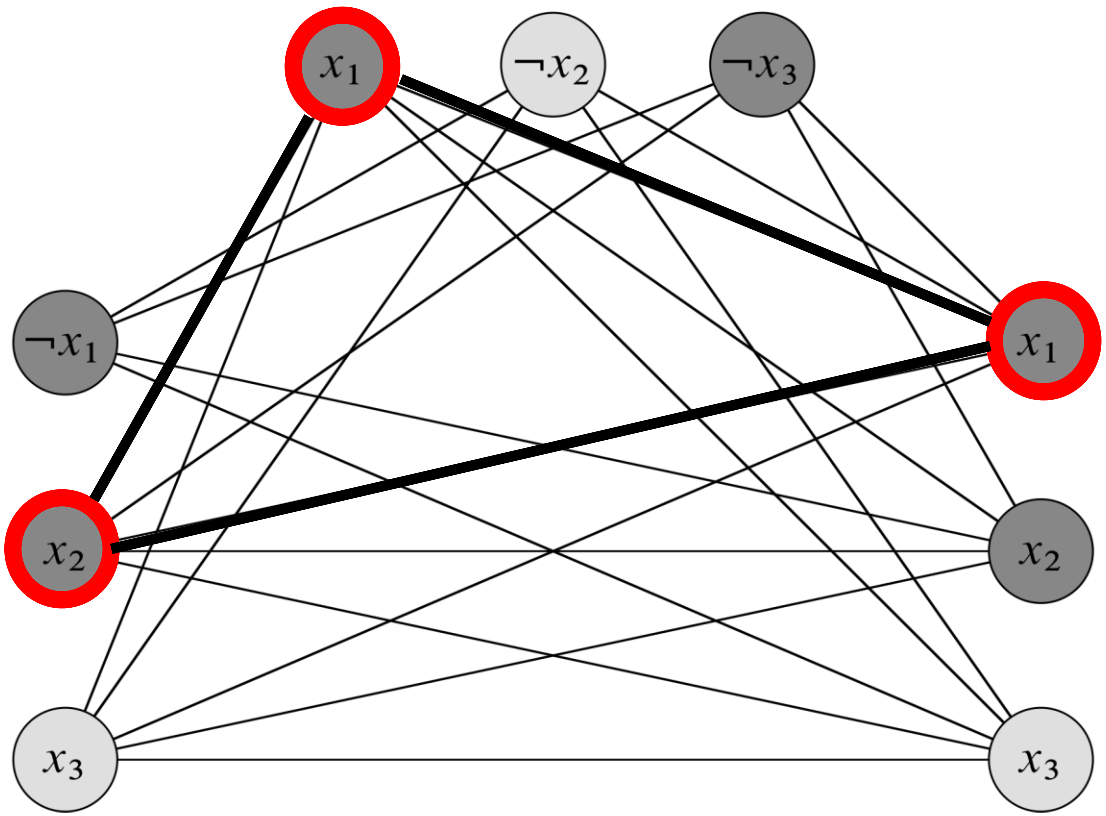
$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$



$C_2 = \neg x_1 \vee x_2 \vee x_3$

$C_3 = x_1 \vee x_2 \vee x_3$

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- Given: S: k-clause 3-CNF formula
- Reduction Algorithm:
  - Compose a graph G of k sets of vertices, each set has three vertices
  - Connect all pairs of vertices (u,v) such that:
    - u and v belong to two different sets
    - If $u=xi$, then $v \neq \neg xi$
  - If there is k-size clique in G, there is a satisfying assignment to S (assign 1 to each vertex in the clique).

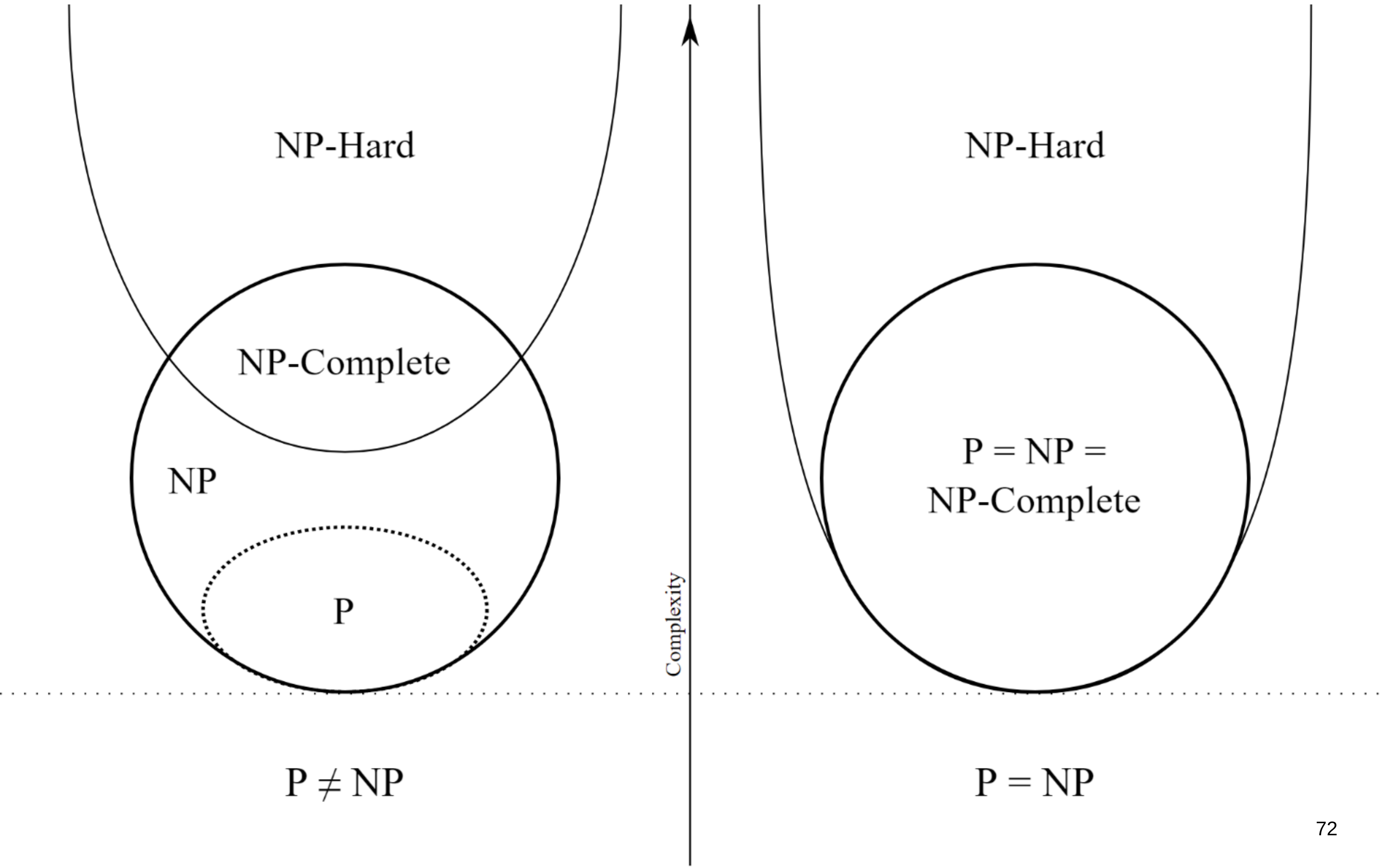
NP-hard Proofs

- To prove B an NP-hard problem:
  - Show a polynomial time reduction algorithm from ONE of the existing NP-hard problems, say B', to B. i.e., B' $\leq_P$ B

NP-Complete Problems

- B is NP-complete problem if:
  1. B $\in$ NP
  2. B is NP-hard

NP-Complete Problems



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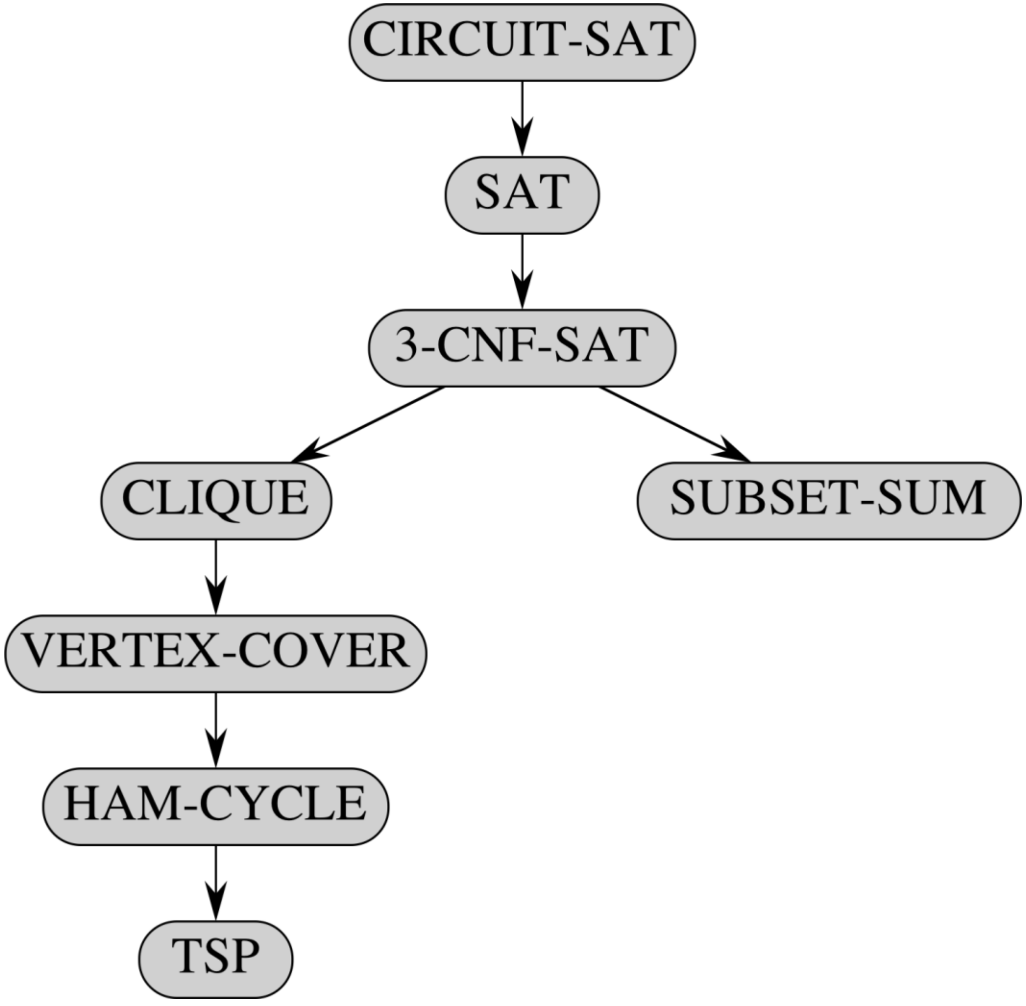
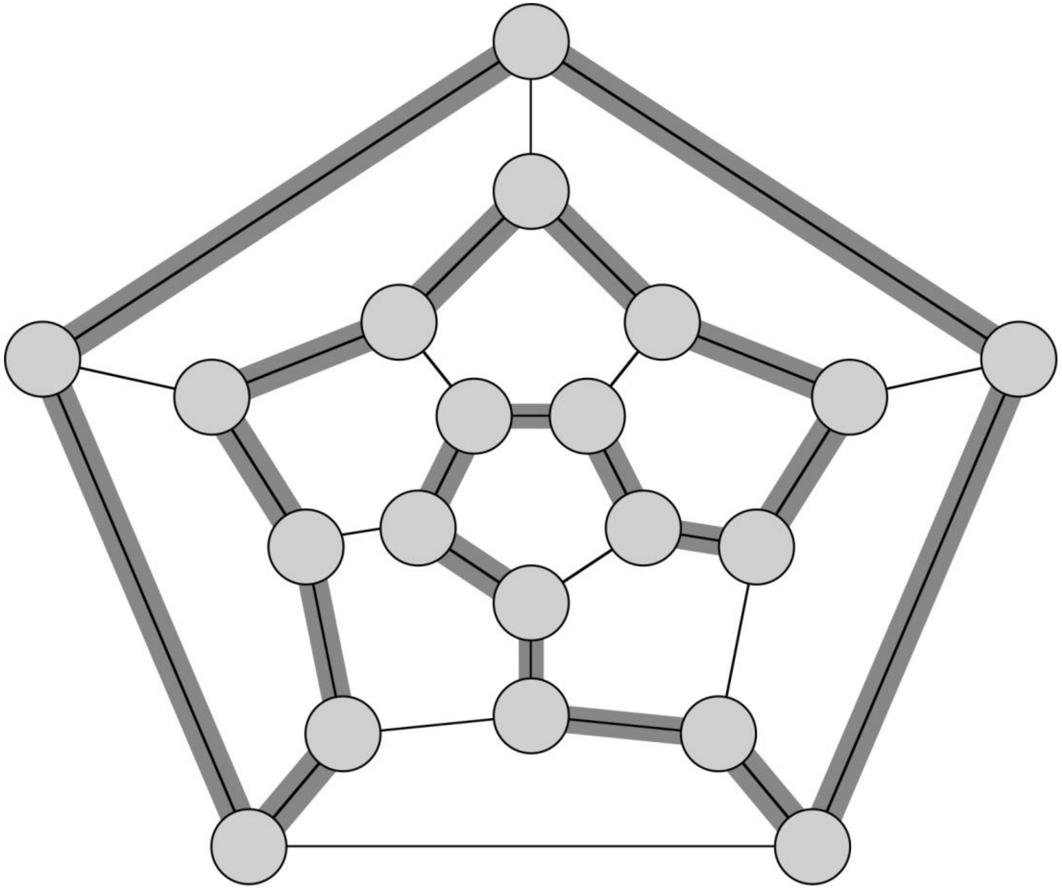


NP-Complete Problems: Examples

- **Hamiltonian Cycle Problem**: Given an undirected or directed graph G, is there a cycle in G that visits each vertex exactly once?



Take Home Messages: Remember?

(1) Computation theory focuses on decision problems

(2) Algorithm complexity is affected by:

(a) the computation model

(b) the input encoding/length

(3) Binary input string (concrete input) is different in length than the algorithm abstract input

Strong vs Weak NP-Completeness

- Abstract input vs Concrete input:
  - Input array of n integers:
    - Abstract input size: a = n (# of integers)
    - Concrete input size in binary: b = n log n (# of bits of the array)

- Weak NP-complete problem:
  - An NP-complete problem that has a known polynomial solution in terms of the abstract input size.

- Strong NP-complete problem:
  - An NP-complete problem that does not have a known polynomial solution in terms of either abstract or concrete input size.

Weak NP-Completeness: Examples

- Subset-Sum Problem:
  - Given set S of n integers and integer T
  - Dynamic Programming solution: O(nT)
  - Abstract input: a_1 = n (integers of S) a_2= 1 (integer T)
  - Concrete input: b_1 = n log n b_2 = log T
  - O(nT) = $O(b_1 2^{b2})$ ➔ exponential in concrete input but polynomial in abstract input ➔ weak NP-complete

- Partition Problem:
  - Given set S of n integers, divide S into two disjoint subsets of equal sum
  - Same solution (and complexity) as Subset-Sum

- 0-1 Knapsack Problem
  - Similar solution to subset-sum (O(nW) for knapsack of weight W)

Weak NP-Completeness

- For weak NP-complete problems, we are able to solve many instances in practical input sizes.

Book Readings

- Ch. 34