Parameterizing a circle

Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors ($\|\mathbf{u}\| = \|\mathbf{v}\| = 1$) that are mutually orthogonal ($\mathbf{u} \cdot \mathbf{v} = 0$) in 3D. Then,

$$\mathbf{p}(t) = \mathbf{c} + R\mathbf{u}\cos t + R\mathbf{v}\sin t$$

is the parameterization of a circle centered at $\mathbf{c}$ with radius R in the plane parallel to $\mathbf{u}$ and $\mathbf{v}$. To see this, note that

- $\mathbf{p}(t)$ lies in a plane containing $\mathbf{c}$ and parallel to $\mathbf{u}$ and $\mathbf{v}$. A plane parallel to $\mathbf{u}$ and $\mathbf{v}$ has normal $\mathbf{n} = \mathbf{u} \times \mathbf{v}$. The rest follows from $(\mathbf{p}(t) - \mathbf{c}) \cdot \mathbf{n} = R(\mathbf{u} \cdot \mathbf{n}) \cos t + R(\mathbf{v} \cdot \mathbf{n}) \sin t = 0$.
- $\mathbf{p}(t)$ lies on a sphere of radius R centered at $\mathbf{c}$. This follows from

$$\begin{aligned}
\|\mathbf{p}(t) - \mathbf{c}\|^2 &= (\mathbf{p}(t) - \mathbf{c}) \cdot (\mathbf{p}(t) - \mathbf{c}) \\
&= (R\mathbf{u}\cos t + R\mathbf{v}\sin t) \cdot (R\mathbf{u}\cos t + R\mathbf{v}\sin t) \\
&= R\mathbf{u}\cos t \cdot (R\mathbf{u}\cos t + R\mathbf{v}\sin t) + R\mathbf{v}\sin t \cdot (R\mathbf{u}\cos t + R\mathbf{v}\sin t) \\
&= R^2(\mathbf{u} \cdot \mathbf{u})\cos^2 t + 2R^2(\mathbf{u} \cdot \mathbf{v})\cos t \sin t + R^2(\mathbf{v} \cdot \mathbf{v})\sin^2 t \\
&= R^2\cos^2 t + R^2\sin^2 t \\
&= R^2
\end{aligned}$$

The intersection of a plane and a sphere is a circle, so $\mathbf{p}(t)$ must represent a circle. Since the plane passes through the sphere's center, the circle and sphere will have the same radius, so the radius of the circle is R.

In the case of the osculating circle, the center ($\mathbf{c} = \mathbf{r}_0 + \kappa_0^{-1}\mathbf{N}_0$) and radius ($R = \kappa_0^{-1}$) can be readily computed. Further, $\mathbf{T}_0$ and $\mathbf{N}_0$ are suitable candidates for $\mathbf{u}$ and $\mathbf{v}$, which makes the osculating circle straightforward to parameterize in the general case once the basic quantities $\mathbf{T}_0 = \mathbf{T}(t_0)$, $\mathbf{N}_0 = \mathbf{N}(t_0)$, and $\kappa_0 = \kappa(t_0)$ are known:

$$\mathbf{p}(t) = \mathbf{r}_0 + \kappa_0^{-1}\cos t\mathbf{T}_0 + \kappa_0^{-1}(1 + \sin t)\mathbf{N}_0.$$

As a point of interest, this is not necessarily the nicest parameterization. I can reparameterize this to

$$\mathbf{p}(t) = \mathbf{r}_0 + \kappa_0^{-1}\sin t\mathbf{T}_0 + \kappa_0^{-1}(1 - \cos t)\mathbf{N}_0,$$

which has the nice properties $\mathbf{p}(0) = \mathbf{r}_0$, $\mathbf{p}'(0) = \kappa_0^{-1}\mathbf{T}_0$, and $\mathbf{p}''(0) = \kappa_0^{-1}\mathbf{N}_0$.