Stirling's Approximation

Math 31B, Fall 2014

Determining the divergence or convergence of the series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2}$$

requires Stirling's approximation,

$$\lim_{n\to\infty} \frac{1}{n!} \left(\frac{n}{e}\right)^n \sqrt{2\pi n} = 1$$

1 Establishing divergence with limits

Let $a_n = \frac{(2n)!}{4^n (n!)^2}$ and $b_n = \frac{1}{\sqrt{n}}$. Then

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\frac{(2n)!}{4^n (n!)^2}}{\frac{1}{\sqrt{n}}} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}(2n)!}{4^n (n!)^2} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}(2n)!}{4^n (n!)^2} \frac{1}{1^2} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}(2n)!}{4^n (n!)^2} \frac{\frac{1}{(2n)!}\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi(2n)}}{\left(\frac{1}{n!}\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^2} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{4^n} \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi(2n)}}{\left(\frac{n}{e}\right)^{2n} (2\pi n)} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{4^n} \frac{(2n)^{2n} \sqrt{2\pi(2n)}}{n^{2n} (2\pi n)} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{4^n} \frac{2^{2n} \sqrt{2\pi(2n)}}{2\pi n} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}\sqrt{2\pi(2n)}}{2\pi n} \\
&= \lim_{n\to\infty} \frac{\sqrt{4\pi}}{2\pi} \\
&= \frac{1}{\sqrt{\pi}}
\end{aligned}$$

Since the limit exists and is finite, a_n and b_n either both converg

2 Establishing Stirling's approximation

I will derive a fairly tight bound on Stirling's approximation using only techniques that we have learned in this course. In fact, it uses many of these techniques: integration and differentiation of logarithms, trapezoid rule, midpoint rule, Taylor series manipulation, alternating series approximations.

Let $r_n = \ln n!$. Then, using a trapezoid rule approximation, which underestimates functions that are concave down,

$$\begin{aligned}
\int_1^n \ln x\,dx &\geq \frac{1}{2}\ln 1 + \sum_{k=2}^{n-1} \ln k + \frac{1}{2}\ln n \\
&= \sum_{k=1}^{n} \ln k - \frac{1}{2}\ln n \\
&= \ln n! - \frac{1}{2}\ln n \\
&= r_n - \frac{1}{2}\ln n \\
r_n &\leq \frac{1}{2}\ln n + \int_1^n \ln x\,dx \\
&= \frac{1}{2}\ln n + [x\ln x - x]_1^n \\
&= \frac{1}{2}\ln n + n\ln n - n - 1\ln 1 + 1 \\
n! = e^{r_n} &\leq \left(\frac{n}{e}\right)^n e\sqrt{n}
\end{aligned}$$

Next, we can use midpoint rule to get another approximation, which this time overestimates functions that are concave down.

$$\begin{aligned}
\int_{\frac{1}{2}}^{n+\frac{1}{2}} \ln x\,dx &\leq \sum_{k=1}^{n} \ln k \\
&= \ln n! \\
&= r_n \\
r_n &\geq \int_{\frac{1}{2}}^{n+\frac{1}{2}} \ln x\,dx \\
&= [x\ln x - x]_{\frac{1}{2}}^{n+\frac{1}{2}} \\
&= \left(n+\frac{1}{2}\right)\ln\left(n+\frac{1}{2}\right) - \left(n+\frac{1}{2}\right) - \frac{1}{2}\ln\frac{1}{2} + \frac{1}{2} \\
&= \left(n+\frac{1}{2}\right)\left(\ln n + \ln\left(1+\frac{1}{2n}\right)\right) - \left(n+\frac{1}{2}\right) - \frac{1}{2}\ln\frac{1}{2} + \frac{1}{2}
\end{aligned}$$

I need a bound on $\ln\left(1+\frac{1}{2n}\right)$, which I can get from what we know about series

$$\begin{aligned}
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \\
\ln\left(1+\frac{1}{2n}\right) &= \frac{1}{2n} - \frac{1}{2}\left(\frac{1}{2n}\right)^2 + \frac{1}{3}\left(\frac{1}{2n}\right)^3 - \frac{1}{4}\left(\frac{1}{2n}\right)^4 + \dots \\
&\geq \frac{1}{2n} - \frac{1}{2}\left(\frac{1}{2n}\right)^2 \qquad \text{(Note: alternating series, decreasing terms)} \\
&= \frac{1}{2n} - \frac{1}{8n^2}
\end{aligned}$$

Finally, I can finish up the lower bound.

$$\begin{aligned}
r_n &\geq \left(n+\frac{1}{2}\right)\left(\ln n + \ln\left(1+\frac{1}{2n}\right)\right) - \left(n+\frac{1}{2}\right) - \frac{1}{2}\ln\frac{1}{2} + \frac{1}{2} \\
&= \left(n+\frac{1}{2}\right)\ln n + \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{2n}\right) - \left(n+\frac{1}{2}\right) - \frac{1}{2}\ln\frac{1}{2} + \frac{1}{2} \\
&\geq \left(n+\frac{1}{2}\right)\ln n + \left(n+\frac{1}{2}\right)\left(\frac{1}{2n} - \frac{1}{8n^2}\right) - \left(n+\frac{1}{2}\right) - \frac{1}{2}\ln\frac{1}{2} + \frac{1}{2} \\
&= n\ln n + \frac{1}{2}\ln n - n + \frac{1}{2}\ln 2 + \frac{1}{2} - \frac{1}{8n} + \frac{1}{4n} - \frac{1}{16n^2} \\
&= n\ln n + \frac{1}{2}\ln n - n + \frac{1}{2}\ln 2 + \frac{1}{2} + \frac{1}{8n} - \frac{1}{16n^2} \\
&= n\ln n + \frac{1}{2}\ln n - n + \frac{1}{2}\ln 2 + \frac{1}{2} + \frac{2n-1}{16n^2} \\
&\geq n\ln n + \frac{1}{2}\ln n - n + \frac{1}{2}\ln 2 + \frac{1}{2} \qquad \text{(Note: } n \geq 1\text{)} \\
n! = e^{r_n} &\geq \left(\frac{n}{e}\right)^n \sqrt{2en}
\end{aligned}$$

Note that we now have the bound

$$\left(\frac{n}{e}\right)^n \sqrt{2en} \leq n! \leq \left(\frac{n}{e}\right)^n e\sqrt{n}$$

To see how tight this bound is,

$$2.33 \leq \sqrt{2e} \leq \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{n}} \leq e \leq 2.72$$

$$0.93 \leq \frac{\sqrt{2e}}{\sqrt{2\pi}} \leq \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \leq \frac{e}{\sqrt{2\pi}} \leq 1.09$$

Note that I have not used the error bounds for trapezoid rule or midpoint rule. The problem with them is that I am doing a very coarse discretization (in fact, $\Delta x = 1$), and their error bounds are accordingly entirely inadequate. Instead, I have used both trapezoid rule and midpoint rule in such a way as to obtain lower and upper bounds. This gives me an effective error bound far stronger than I could have obtained from the error bounds in the book.

3 Establishing divergence with bounds

Although these bounds are not as strong in some sense as the approximation used at the beginning, they are more than strong enough to establish divergence of the original series. In this case, I have bounds rather than a limit, so I will accordingly use the comparison test instead of the limit comparison test.

Let $a_n = \frac{(2n)!}{4^n(n!)^2}$.

$$\begin{aligned}
a_n &= \frac{(2n)!}{4^n(n!)^2} \\
&\geq \frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{2e(2n)}}{4^n(n!)^2} && \text{Note: use lower bound in numerator} \\
&\geq \frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{2e(2n)}}{4^n\left(\left(\frac{n}{e}\right)^n e\sqrt{n}\right)^2} && \text{Note: use upper bound in denominator} \\
&= \frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{2e(2n)}}{4^n\left(\frac{n}{e}\right)^{2n} e^2 n} \\
&= \frac{(2n)^{2n}\sqrt{4en}}{4^n n^{2n} e^2 n} \\
&= \frac{2^{2n}\sqrt{4en}}{4^n e^2 n} \\
&= \frac{\sqrt{4en}}{e^2 n} \\
&= \frac{2\sqrt{e}}{e^2\sqrt{n}} = b_n \qquad \text{(definition of } b_n\text{.)}
\end{aligned}$$

The series $\displaystyle\sum_{n=1}^{\infty} b_n = \frac{2\sqrt{e}}{e^2}\sum_{n=1}^{\infty} n^{-\frac{1}{2}}$ is a divergent p-series, so the original series diverges by the comparison test.

4 Alternating version

Determining the divergence or convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{4^n(n!)^2}$$

requires showing that the series is alternating, and has terms that are decreasing monotonically to zero. The fact that it is alternating is obvious. The fact that its terms are decreasing in magnitude can be shown directly

$$\begin{aligned}
|a_n| - |a_{n+1}| &= \frac{(2n)!}{4^n(n!)^2} - \frac{(2n+2)!}{4^{n+1}((n+1)!)^2} \\
&= \frac{1}{((n+1)!)^2}\left(\frac{(2n)!(n+1)^2}{4^n} - \frac{(2n+2)!}{4^{n+1}}\right) \\
&= \frac{1}{((n+1)!)^2 4^{n+1}}(4(2n)!(n+1)^2 - (2n+2)!) \\
&= \frac{(2n)!}{((n+1)!)^2 4^{n+1}}(4(n+1)^2 - (2n+2)(2n+1)) \\
&= \frac{2(n+1)(2n)!}{((n+1)!)^2 4^{n+1}}(2(n+1) - (2n+1)) \\
&= \frac{2(n+1)(2n)!}{((n+1)!)^2 4^{n+1}} > 0
\end{aligned}$$

The terms go to zero proportional to $n^{-\frac{1}{2}}$ as shown in the positive case. The series converges by the Leibniz test. Since the corresponding non-alternating series diverges, this series converges conditionally.