Midterm 2

Name: ______________________________ ID: ______________ Section: ____

Problem	1	2	3	4	5	6	7	8	9	total
Score										

You have 50 minutes to complete this midterm. You must show your work to receive credit. This exam contains 9 questions worth a total of 60 points.

Problem 1 (10 points): **Evaluate:** $\displaystyle\int \frac{e^x}{(e^{2x}+1)^2}\,dx$

$$\begin{aligned}
\int \frac{e^x}{(e^{2x}+1)^2}\,dx &= \int \frac{du}{(u^2+1)^2} && (u = e^x, du = e^x\,dx) \\
&= \int \frac{\sec^2 v\,dv}{(\tan^2 v+1)^2} && (u = \tan v, du = \sec^2 v\,dv) \\
&= \int \frac{\sec^2 v\,dv}{\sec^4 v} \\
&= \int \cos^2 v\,dv \\
&= \int \frac{1+\cos(2v)}{2}\,dv \\
&= \frac{1}{2}v + \frac{1}{4}\sin(2v) + C \\
&= \frac{1}{2}v + \frac{1}{2}\sin v\cos v + C \\
&= \frac{1}{2}\tan^{-1} u + \frac{\tan v}{2\sec^2 v} + C \\
&= \frac{1}{2}\tan^{-1} u + \frac{u}{2(u^2+1)} + C \\
&= \frac{1}{2}\tan^{-1} e^x + \frac{e^x}{2(e^{2x}+1)} + C
\end{aligned}$$

Problem 2 (10 points): Derive a recurrence relation that allows the integral $\int \sec^n x\,dx$ to be evaluated once the integral $\int \sec^{n-2} x\,dx$ is known. You may assume $n \geq 2$.

This is done with integration by parts with

$$\begin{aligned}
u &= \sec^{n-2} x \\
v' &= \sec^2 x \\
u' &= (n-2)\sec^{n-3} x(\tan x \sec x) \\
&= (n-2)\tan x \sec^{n-2} x \\
v &= \tan x
\end{aligned}$$

Then,

$$\begin{aligned}
\int \sec^n x\,dx &= \tan x \sec^{n-2} x - \int (n-2)\tan x \sec^{n-2} x \tan x\,dx \\
&= \tan x \sec^{n-2} x - (n-2)\int \tan^2 x \sec^{n-2} x\,dx \\
&= \tan x \sec^{n-2} x - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x\,dx \\
&= \tan x \sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx \\
(n-1)\int \sec^n x\,dx &= \tan x \sec^{n-2} x + (n-2)\int \sec^{n-2} x\,dx \\
\int \sec^n x\,dx &= \frac{1}{n-1}\tan x \sec^{n-2} x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx
\end{aligned}$$

Problem 3 (10 points): Evaluate $\displaystyle\int_1^\infty \frac{dx}{x^3+x}$.

$$
\begin{aligned}
\frac{1}{x^3+x} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\
1 &= A(x^2+1) + (Bx+C)x \\
1 &= (A+B)x^2 + Cx + A \\
A &= 1 \\
C &= 0 \\
B &= -1 \\
\int \frac{dx}{x^3+x} &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} \\
&= \ln|x| - \frac{1}{2}\int \frac{du}{u} \qquad (u = x^2+1, du = 2x\,dx) \\
&= \ln|x| - \frac{1}{2}\ln|u| + C \\
&= \ln|x| - \frac{1}{2}\ln|x^2+1| + C \\
\int_1^\infty \frac{dx}{x^3+x} &= \lim_{R\to\infty}\int_1^R \frac{dx}{x^3+x} \\
&= \lim_{R\to\infty}\left[\ln|x| - \frac{1}{2}\ln|x^2+1|\right]_1^R \\
&= -\left(\ln|1| - \frac{1}{2}\ln|2|\right) + \lim_{R\to\infty}\left(\ln|R| - \frac{1}{2}\ln|R^2+1|\right) \\
&= \frac{1}{2}\ln 2 + \frac{1}{2}\lim_{R\to\infty}\big(2\ln R - \ln(R^2+1)\big) \\
&= \frac{1}{2}\ln 2 + \frac{1}{2}\lim_{R\to\infty}\ln\left(\frac{R^2}{R^2+1}\right) \\
&= \frac{1}{2}\ln 2 + \frac{1}{2}\ln\lim_{R\to\infty}\left(\frac{R^2}{R^2+1}\right) \\
&= \frac{1}{2}\ln 2 + \frac{1}{2}\ln 1 \\
&= \frac{1}{2}\ln 2
\end{aligned}
$$

Problem 4 (10 points): **Find the arclength of $y = x^2$ in the interval $x \in [-1, 1]$.**

$$
\begin{aligned}
\int \sqrt{1+(y')^2}\,dx &= \int \sqrt{1+(2x)^2}\,dx \\
&= \frac{1}{2}\int \sec^2 u \sqrt{1+\tan^2 u}\,dx \qquad\qquad (2x = \tan u, 2dx = \sec^2 u\,du) \\
&= \frac{1}{2}\int \sec^3 u\,dx \\
&= \frac{1}{2}\left(\frac{1}{2}\tan u \sec u + \frac{1}{2}\int \sec u\,du\right) \\
&= \frac{1}{4}\tan u \sec u + \frac{1}{4}\ln|\sec u + \tan u| + C \\
&= \frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln|\sqrt{1+4x^2} + 2x| + C \\
\int_{-1}^{1} \sqrt{1+(y')^2}\,dx &= \left[\frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln|\sqrt{1+4x^2} + 2x|\right]_{-1}^{1} \\
&= \frac{1}{2}\sqrt{1+4} + \frac{1}{4}\ln|\sqrt{1+4}+2| + \frac{1}{2}\sqrt{1+4} - \frac{1}{4}\ln|\sqrt{1+4}-2| \\
&= \sqrt{5} + \frac{1}{4}\ln(\sqrt{5}+2) - \frac{1}{4}\ln(\sqrt{5}-2) \\
&= \sqrt{5} + \frac{1}{4}\ln\left(\frac{\sqrt{5}+2}{\sqrt{5}-2}\right) \\
&= \sqrt{5} + \frac{1}{4}\ln\left(\frac{(\sqrt{5}+2)^2}{(\sqrt{5}-2)(\sqrt{5}+2)}\right) \\
&= \sqrt{5} + \frac{1}{4}\ln\left((\sqrt{5}+2)^2\right) \\
&= \sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)
\end{aligned}
$$

The trig integral was evaluated using the recurrence derived earlier on the midterm.

Problem 5 (4 points): **Determine (with justification) the convergence or divergence of the series $\sum_{n=0}^{\infty} e^{-n}\sin^2(n^2)$.**

This series converges since $0 \le e^{-n}\sin^2(n^2) \le e^{-n}$ and

$$\sum_{n=0}^{\infty} e^{-n}\sin^2(n^2) \le \sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-e^{-1}}$$

Problem 6 (4 points): Determine (with justification) the convergence or divergence of the series $\sum_{n=2}^{\infty} \frac{1+e^{-2n}}{n \ln^3 n}$**.**

This series converges since

$$0 \le \frac{1+e^{-2n}}{n \ln^3 n} \le \frac{2}{n \ln^3 n}$$

and the corresponding integral

$$\int_2^{\infty} \frac{2\,dn}{n \ln^3 n} = \lim_{R \to \infty} \int_2^{R} \frac{2\,dn}{n \ln^3 n} = \lim_{R \to \infty} \left[-\frac{1}{\ln^2 n} \right]_2^R = \lim_{R \to \infty} \left(\frac{1}{\ln^2 2} - \frac{1}{\ln^2 R} \right) = \frac{1}{\ln^2 2}$$

converges.

Problem 7 (4 points): Determine (with justification) the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

This series diverges since

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} = \sum_{n=1}^{2} \frac{\ln n}{n} + \sum_{n=3}^{\infty} \frac{\ln n}{n} \geq \sum_{n=1}^{2} \frac{\ln n}{n} + \sum_{n=3}^{\infty} n^{-1}$$

is a divergent p-series.

Problem 8 (4 points): Determine (with justification) the convergence or divergence of the series $\sum_{n=0}^{\infty} \frac{n!}{e^n}$.

This series diverges because the terms do not converge to zero. In fact, as $n \to \infty$, we have $\frac{n!}{e^n} \to \infty$. Determining this requires knowing that $n!$ grows faster than e^n. One way to see this is to note that

$$(n!)^2 = \left(\prod_{k=1}^{n} k\right)^2 = \left(\prod_{k=1}^{n} k\right)\left(\prod_{k=1}^{n}(n-k+1)\right) = \prod_{k=1}^{n} k(n-k+1) \geq \prod_{k=1}^{n} n = n^n$$

noting that $k(n-k+1) \geq n$ is true when $0 \leq kn-k^2+k-n = -(n-k)(1-k)$ or $1 \leq k \leq n$. Then, $n! \geq (\sqrt{n})^n$. Finally,

$$\lim_{n\to\infty} \frac{n!}{e^n} \geq \lim_{n\to\infty} \frac{(\sqrt{n})^n}{e^n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{e}\right)^n = \infty^\infty = \infty$$

Problem 9 (4 points): Determine (with justification) the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n^3+1}}$.

This series converges, since

$$\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n^3+1}} \leq \sum_{n=1}^{\infty} \frac{2}{\sqrt{n^3+1}} < \sum_{n=1}^{\infty} \frac{2}{\sqrt{n^3}} = 2\sum_{n=1}^{\infty} n^{-\frac{3}{2}}$$

is a convergent p-series.