Midterm 1

Name: ______________________________ ID: ______________ Section: ____

You have 50 minutes to complete this quiz. You must show your work to receive credit. There are more problems on the back.

Problem 1 (10 points): Evaluate: $\int e^{-x}\sin x\,dx$ and $\int e^{-x}\cos x\,dx$.

Use integration by parts twice to get one of them.

$$\begin{aligned}
\int e^{-x}\sin x\,dx &= e^{-x}(-\cos x) - \int (-e^{-x})(-\cos x)\,dx \\
&= -e^{-x}\cos x - \int e^{-x}\cos x\,dx \\
&= -e^{-x}\cos x - e^{-x}\sin x + \int (-e^{-x})\sin x\,dx \\
\int e^{-x}\sin x\,dx &= -\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x + C
\end{aligned}$$

Then substitute back to get the other.

$$\begin{aligned}
\int e^{-x}\cos x\,dx &= -e^{-x}\cos x - \int e^{-x}\sin x\,dx \\
&= -e^{-x}\cos x - \left(-\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x + C\right) \\
&= -\frac{1}{2}e^{-x}\cos x + \frac{1}{2}e^{-x}\sin x + C_2
\end{aligned}$$

Problem 2 (10 points): Evaluate: $\int xe^{-x}\sin x\,dx$.

Use integration by parts again, reusing the results of previous problems.

$$u = x \qquad du = dx \qquad dv = e^{-x}\sin x\,dx \qquad v = -\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x$$

$$\int xe^{-x}\sin x\,dx = x\left(-\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x\right) - \int\left(-\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x\right)dx$$
$$= -\frac{1}{2}xe^{-x}\cos x - \frac{1}{2}xe^{-x}\sin x + \frac{1}{2}\int e^{-x}\cos x\,dx + \frac{1}{2}\int e^{-x}\sin x\,dx$$
$$\int xe^{-x}\sin x\,dx = -\frac{1}{2}xe^{-x}\cos x - \frac{1}{2}xe^{-x}\sin x - \frac{1}{2}e^{-x}\cos x + C$$

Problem 3 (10 points): **Evaluate:** $\lim_{x\to 0^+} \frac{x^{2x}-1}{x\ln x}$

The denominator tends to zero. To see if the numerator does, I need to know $\lim_{x\to 0^+} x^{2x}$.

$$\ln\left(\lim_{x\to 0^+} x^{2x}\right) = \lim_{x\to 0^+}\ln(x^{2x}) = \lim_{x\to 0^+} 2x\ln x = 2\lim_{x\to 0^+}\frac{\ln x}{x^{-1}} = 2\lim_{x\to 0^+}\frac{x^{-1}}{-x^{-2}} = -2\lim_{x\to 0^+} x = 0$$

Thus, $\lim_{x\to 0^+} x^{2x} = e^0 = 1$, and the numerator is also zero. Now I can apply L'Hôpital's rule, which means I must now differentiate x^{2x}, which I can do with logarithmic differentiation.

$$y = x^{2x}$$
$$\ln y = \ln(x^{2x}) = 2x\ln x$$
$$\frac{y'}{y} = 2\ln x + 2\frac{x}{x} = 2(\ln x + 1)$$
$$y' = 2(\ln x + 1)x^{2x}$$

From here, L'Hôpital's rule finishes things up pretty quickly.

$$\lim_{x\to 0^+}\frac{x^{2x}-1}{x\ln x} = \lim_{x\to 0^+}\frac{2(\ln x+1)x^{2x}}{\ln x + 1} = 2\lim_{x\to 0^+} x^{2x} = 2$$

Problem 4 (10 points): **For a falling object of mass m, free-fall with air resistance can be modeled with $v' = -\frac{k}{m}\left(v + \frac{mg}{k}\right)$. The object starts at rest and tends towards a terminal velocity v_1. How long does it take the object to reach half its terminal velocity? (Note that $v_1 < 0$, since the object is falling.)**

At terminal velocity, $v' = 0$, so that $v_1 = -\frac{mg}{k}$, or $k = -\frac{mg}{v_1}$. The object's velocity will be $v = Ce^{-\frac{k}{m}t} - \frac{mg}{k}$. Since $v(0) = 0$, $C = \frac{mg}{k}$. Finally,

$$v = \left(e^{-\frac{k}{m}t} - 1\right)\frac{mg}{k}$$
$$= -\left(e^{\frac{g}{v_1}t} - 1\right)v_1$$

The time T required to reach half terminal velocity $\frac{v_1}{2}$ is

$$\begin{aligned} -\left(e^{\frac{g}{v_1}T} - 1\right)v_1 &= \frac{v_1}{2} \\ e^{\frac{g}{v_1}T} - 1 &= -\frac{1}{2} \\ e^{\frac{g}{v_1}T} &= \frac{1}{2} \\ \frac{g}{v_1}T &= -\ln 2 \\ T &= -\frac{v_1 \ln 2}{g} \end{aligned}$$

Problem 5 (10 points): **Evaluate:** $\int_0^\pi \frac{\sin x}{2\cos x + 3}\, dx$

Use the substitution $u = 2\cos x + 3$, $du = -2\sin x\, dx$. $x = 0 \implies u = 2\cos 0 + 3 = 5$. $x = \pi \implies u = 2\cos\pi + 3 = 1$.

$$\int_0^\pi \frac{\sin x}{2\cos x + 3}\, dx = -\frac{1}{2}\int_5^1 \frac{du}{u} = -\frac{1}{2}\Big[\ln|u|\Big]_5^1 = -\frac{1}{2}(\ln 1 - \ln 5) = \frac{1}{2}\ln 5$$

Problem 6 (10 points): **Given the function $f(x) = xe^x$. (2 points each)**

(a) Identify the critical points.

(b) Use the second derivative test to identify each critical point as a local minimum or local maximum.

(c) Identify the inflection points.

(d) Evaluate the limits $\lim_{x\to 0} f(x)$, $\lim_{x\to\infty} f(x)$, and $\lim_{x\to-\infty} f(x)$.

(e) Use this information to sketch the function.

First, lets compute the derivatives.

$$f'(x) = \frac{d}{dx}(xe^x) = xe^x + e^x = (x+1)e^x$$

$$f''(x) = \frac{d}{dx}((x+1)e^x) = (x+1)e^x + e^x = (x+2)e^x$$

(a) We require $f'(x) = 0$. Since $e^x > 0$, the only critical point is $x = -1$.

(b) Since $f''(-1) = ((-1)+2)e^{-1} = e^{-1} > 0$, the critical point is a local minimum.

(c) We require $f''(x) = 0$. The only inflection point is $x = -2$.

(d) The first limit is $\lim_{x\to 0} xe^x = (0)e^0 = 0$, and the second limit is $\lim_{x\to\infty} xe^x = \infty$. The third limit can be evaluated with L'Hôpital's rule.

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^x = 0.$$

(e) The function is, with some of its features labeled,