Final

Name: ____________________________________ ID: ________________ Section: ____

Problem	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	total
Score																

You have 180 minutes to complete this final. You must show your work to receive credit. This exam contains 15 questions.

Problem 1 (5 points): Find the sum $\sum_{n=0}^{\infty} e^{-2n}$.

This is a geometric series.

$$\sum_{n=0}^{\infty} e^{-2n} = \sum_{n=0}^{\infty} \left(\frac{1}{e^2}\right)^n = \frac{1}{1-e^{-2}}$$

Problem 2 (5 points): Determine (with justification) whether the series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$ converges, or diverges.

Using the ratio test,

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{(1+n^{-1})^2}{(2+2n^{-1})(2+n^{-1})} \right| \\
&= \frac{1}{4}
\end{aligned}$$

Since $\rho < 1$, the series converges absolutely.

Problem 3 (5 points): Determine (with justification) whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^4 - n - 1}}$ converges, or diverges.

Let $a_n = \dfrac{1}{\sqrt{n^4 - n - 1}}$ and $b_n = n^{-2}$. Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^4-n-1}}}{n^{-2}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4 - n - 1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 - n^{-3} - n^{-4}}} = 1.$$

The series $\displaystyle\sum_{n=2}^{\infty} b_n$ converges absolutely (convergent p-series), so the original converges by the limit comparison test.

Problem 4 (5 points): Determine (with justification) whether the following series S converges absolutely, converges conditionally, or diverges:

$$S = \frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \dots$$

The series $|a_n| = \frac{1}{n}$ is a divergent p-series, so the series cannot converge absolutely. Note that the terms of the series can be grouped:

$$S = \left(\frac{1}{1} + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) - \left(\frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10}\right) - \left(\frac{1}{11} + \frac{1}{12}\right) + \dots$$

In this form, the series is alternating and has monotonically decreasing terms, so it must converge. The series converges conditionally.

Problem 5 (5 points): Determine the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1}$**.**

Using the ratio test,

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}x^{n+1}}{n+2}}{\frac{2^n x^n}{n+1}} \right| \\
&= 2|x| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| \\
&= 2|x|
\end{aligned}$$

If $2|x| < 1$, convergence is absolute. If $2|x| > 1$, the series diverges. Next, we need to determine the endpoints, $x = \pm\frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{2^n}{n+1}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \qquad\qquad \sum_{n=0}^{\infty} \frac{2^n}{n+1}\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

The first series is a divergent p-series. The second is an alternating series with decreasing terms and thus converges. The interval of convergence is $[-\frac{1}{2}, \frac{1}{2})$.

Problem 6 (Extra Credit[1]: 10 points): **Determine the function** $f(x)$ **whose Taylor series is** $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1}$ **and use it to evaluate the series** $\displaystyle\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}$**.**

$$
\begin{aligned}
\sum_{n=0}^{\infty} z^n &= \frac{1}{1-z} \\
\int \left(\sum_{n=0}^{\infty} z^n \right) dz &= \int \frac{dz}{1-z} \\
\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} &= -\ln|1-z| \\
\sum_{n=0}^{\infty} \frac{z^n}{n+1} &= -\frac{\ln|1-z|}{z} \\
\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1} &= -\frac{\ln|1-2x|}{2x} \\
\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1} &= -\frac{\ln\left|1-2\left(\frac{1}{4}\right)\right|}{2\left(\frac{1}{4}\right)} \\
&= 2\ln 2
\end{aligned}
$$

[1]Making this question extra credit helps students using the final for their midterm grade. It makes little difference when computing the final grade, since the final result will be curved.

Problem 7 (10 points): **Evaluate:** $\lim_{x\to 0^+} \frac{x^x - 1}{x \ln x}$

Note that this question is virtually identical to a problem on the first midterm and can be solved in exactly the same way. The denominator tends to zero since

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = -\lim_{x\to 0^+} x = 0$$

Next, I need

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x \ln x} = e^{\lim\limits_{x\to 0^+} x \ln x} = e^0 = 1$$

This shows that the denominator also tends to zero. Now I can apply L'Hôpital's rule.

$$\begin{aligned}
\lim_{x\to 0^+} \frac{x^x - 1}{x \ln x} &= \lim_{x\to 0^+} \frac{\frac{d}{dx}(x^x - 1)}{\frac{d}{dx}(x \ln x)} \\
&= \lim_{x\to 0^+} \frac{\frac{d}{dx} e^{x \ln x}}{\ln x + 1} \\
&= \lim_{x\to 0^+} \frac{e^{x \ln x} \frac{d}{dx}(x \ln x)}{\ln x + 1} \\
&= \lim_{x\to 0^+} \frac{e^{x \ln x}(\ln x + 1)}{\ln x + 1} \\
&= \lim_{x\to 0^+} e^{x \ln x} \\
&= 1
\end{aligned}$$

Problem 8 (10 points): The differential equation $y'' + 3y' + 2y = 0$ has the general solution $y = Ae^{at} + Be^{bt}$, where $a \neq b$ and the scalars A and B are arbitrary. Find a and b (the choice is not unique; pick one). The scalars A and B depend on the initial conditions. Use the conditions $y(0) = 0$ and $y'(0) = 1$ to find A and B.

$$
\begin{aligned}
y &= Ae^{at} + Be^{bt} \\
y' &= Aae^{at} + Bbe^{bt} \\
y'' &= Aa^2e^{at} + Bb^2e^{bt} \\
0 &= y'' + 3y' + 2y \\
&= (Aa^2e^{at} + Bb^2e^{bt}) + 3(Aae^{at} + Bbe^{bt}) + 2(Ae^{at} + Be^{bt}) \\
&= Ae^{at}(a^2 + 3a + 2) + Be^{bt}(b^2 + 3b + 2)
\end{aligned}
$$

Since this must hold for any A and B, it must hold when $A = 1$ and $B = 0$, so that $0 = a^2+3a+2 = (a+1)(a+2)$. It must then also be true that $0 = b^2+3b+2 = (b+1)(b+2)$. Lets choose $a = -1$ and $b = -2$. (The choice $a = -2$ and $b = -1$ is also okay.)

$$
\begin{aligned}
y &= Ae^{-t} + Be^{-2t} \\
y(0) &= A + B = 0 \\
B &= -A \\
y' &= -Ae^{-t} - 2Be^{-2t} \\
y'(0) &= -A - 2B = 1 \\
A &= 1 \\
B &= -1 \\
y &= e^{-t} - e^{-2t}
\end{aligned}
$$

Problem 9 (10 points): **Integrate** $\displaystyle\int \frac{dx}{x^2(1-x^2)}$.

$$\begin{aligned}
\frac{1}{x^2(1-x^2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1} \\
-1 &= Ax(x+1)(x-1) + B(x+1)(x-1) + Cx^2(x-1) + Dx^2(x+1) \\
-1 &= B(1)(-1) \qquad (x=0) \\
B &= 1 \\
-1 &= D1^2(2) \qquad (x=1) \\
D &= -\frac{1}{2} \\
-1 &= C(-1)^2(-2) \qquad (x=-1) \\
C &= \frac{1}{2} \\
-1 &= A(2)(3)(1) + (1)(3)(1) + \frac{1}{2}(2^2)(1) - \frac{1}{2}(2^2)(3) \qquad (x=2) \\
A &= 0 \\
\int \frac{dx}{x^2(1-x^2)} &= \int \frac{dx}{x^2} + \frac{1}{2}\int \frac{du}{x+1} - \frac{1}{2}\int \frac{dx}{x-1} \\
&= -x^{-1} + \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C
\end{aligned}$$

Problem 10 (10 points): Show that the integral $\int \csc^2 x \sec x \, dx$ can be converted into $\int \frac{du}{u^2(1-u^2)}$ using a substitution.

$$\begin{aligned}
\int \csc^2 x \sec x \, dx &= \int \frac{1}{\sin^2 x \cos x} \, dx \\
&= \int \frac{\cos x}{\sin^2 x \cos^2 x} \, dx \\
&= \int \frac{\cos x}{\sin^2 x (1 - \sin^2 x)} \, dx \\
&= \int \frac{du}{u^2(1-u^2)} \qquad u = \sin x, du = \cos x \, dx
\end{aligned}$$

Problem 11 (10 points): Find the surface area of the solid obtained by revolving $y = \sqrt{x}$ about the x axis in the interval $[0, 1]$.

$$A = 2\pi \int_0^1 y\sqrt{1 + (y')^2}\, dx$$

$$A = 2\pi \int_0^1 \sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$$

$$A = 2\pi \int_0^1 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx$$

$$A = 2\pi \int_0^1 \sqrt{x + \frac{1}{4}}\, dx$$

$$A = 2\pi \int_{\frac{1}{4}}^{\frac{5}{4}} \sqrt{u}\, du \qquad u = x + \frac{1}{4}, du = dx$$

$$A = 2\pi \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{\frac{1}{4}}^{\frac{5}{4}}$$

$$A = \frac{4}{3}\pi\left(\left(\frac{5}{4}\right)^{\frac{3}{2}} - \left(\frac{1}{4}\right)^{\frac{3}{2}}\right)$$

$$A = \frac{\pi}{6}(5\sqrt{5} - 1)$$

Problem 12 (10 points): Find the length of the curve $y = \ln(\sec x)$ in the interval $x \in [0, \frac{\pi}{6}]$.

$$\begin{aligned}
L &= \int_0^{\frac{\pi}{6}} \sqrt{1 + (y')^2}\, dx \\
&= \int_0^{\frac{\pi}{6}} \sqrt{1 + \left(\frac{d}{dx} \ln(\sec x)\right)^2}\, dx \\
&= \int_0^{\frac{\pi}{6}} \sqrt{1 + \tan^2 x}\, dx \\
&= \int_0^{\frac{\pi}{6}} \sec x\, dx \\
&= [\ln|\sec x + \tan x|]_0^{\frac{\pi}{6}} \\
&= \ln\left|\sec\frac{\pi}{6} + \tan\frac{\pi}{6}\right| - \ln|\sec 0 + \tan 0| \\
&= \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| \\
&= \ln\left|\frac{3}{\sqrt{3}}\right| \\
&= \frac{\ln 3}{2}
\end{aligned}$$

Problem 13 (10 points): Integrate $\int_0^2 x\ln(x+1)\,dx$.

$$\begin{aligned}
\int_0^2 x\ln(x+1)\,dx &= \left[\frac{x^2}{2}\ln(x+1)\right]_0^2 - \int_0^2 \frac{x^2}{2}\frac{1}{x+1}\,dx \\
&= 2\ln 3 - \frac{1}{2}\int_0^2 \frac{x^2}{x+1}\,dx \\
&= 2\ln 3 - \frac{1}{2}\int_1^3 \frac{(u-1)^2}{u}\,dx \qquad x+1=u, dx=du \\
&= 2\ln 3 - \frac{1}{2}\int_1^3 u-2+u^{-1}\,dx \\
&= 2\ln 3 - \frac{1}{2}\left[\frac{u^2}{2}-2u+\ln|u|\right]_1^3 \\
&= 2\ln 3 - \frac{1}{2}\left(\frac{3^2}{2}-2(3)+\ln|3|-\frac{1^2}{2}+2(1)-\ln|1|\right) \\
&= 2\ln 3 - \frac{1}{2}\ln 3 \\
&= \frac{3}{2}\ln 3
\end{aligned}$$

Problem 14 (10 points): Derive a recurrence relation relating $\Gamma(z+1)$ and $\Gamma(z)$, where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx$ is the gamma function and $z > 0$. You may assume $\lim_{x\to\infty} x^n e^{-x} = 0$ for any real number n.

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty x^z e^{-x}\,dx \\
&= \left[-x^z e^{-x}\right]_0^\infty - \int_0^\infty (zx^{z-1})(-e^{-x})\,dx \\
&= z\int_0^\infty x^{z-1}e^{-x}\,dx \\
&= z\Gamma(z)
\end{aligned}$$

Problem 15 (Extra Credit[2]: 10 points): Recall that atmospheric pressure decays exponentially with height. Find the pressure on a wall of a building that has width w and height h. The atmospheric pressure at the bottom of the building is measured to be p_0, and the atmospheric pressure at the top of the building is measured to be p_1.

Let the pressure be $p(y) = ae^{ry}$.

$$\begin{aligned} p_0 &= p(0) = a \\ p_1 &= p(h) = ae^{rh} \\ e^{rh} &= \frac{p_1}{p_0} \\ rh &= \ln p_1 - \ln p_0 \\ r &= \frac{\ln p_1 - \ln p_0}{h} \end{aligned}$$

Next, consider a thin strip from the wall of the building. The strip i has height Δy_i and length w, so its area is $A_i = w\Delta y_i$. The pressure on this strip is $p(y_i) = p_0 e^{ry_i}$. The force is then

$$\begin{aligned} F &\approx \sum_i F_i \\ &= \sum_i p(y_i) A_i \\ &= \sum_i p_0 e^{ry_i} w \Delta y_i \\ F &= \int_0^h p_0 e^{ry} w \, dy \\ &= p_0 w \int_0^h e^{ry} \, dy \\ &= p_0 w \left[\frac{e^{ry}}{r} \right]_0^h \\ &= p_0 w \left(\frac{e^{rh}}{r} - \frac{1}{r} \right) \\ &= \frac{p_0 w (e^{rh} - 1)}{r} \\ &= \frac{p_0 w (\frac{p_1}{p_0} - 1)}{\frac{\ln p_1 - \ln p_0}{h}} \\ &= \frac{wh(p_1 - p_0)}{\ln p_1 - \ln p_0} \end{aligned}$$

[2]Making this question extra credit helps students using the final for their midterm grade. It makes little difference when computing the final grade, since the final result will be curved.