# Atmospheric Pressure Profile

Math 31B, Fall 2014

For a fluid that is incompressible, the density $\rho$ will be constant with respect to changes in pressure. When this occurs, the pressure at depth $h$ is just $\rho g h$. This works well for water, but it does not work well for gasses. For the atmosphere, the density of air will depend on the pressure applied to it. In this case, the density profile will not be linear. The goal of this document is to work out what the profile will be.

To do this, I will need to make many assumptions. I will assume that the air is ideal and composed of a singe type of gas (e.g., nitrogen). I will assume that the temperature does not depend on altitude or location and that it is stationary (no wind). Some of these are reasonable (ideal gas), but others are clearly suspect (constant temperature).

## 1 Pressure change with elevation

For the purposes of this derivation, I will assume that I am interested in a column of air. The column has cross section area $A$. The density at any altitude $y$ is $\rho(y)$. Similarly, the pressure at any altitude is $p(y)$.

Lets consider next the mass of the air column. The air in the altitude range $[y, y + \Delta y]$ has approximate density $\rho(y)$. The volume of this air is $A\Delta y$, so its mass is $\rho(y)A\Delta y$. Adding these masses up for all of the air above some height $y$ gives a mass

$$m(y) = A \int_y^\infty \rho(z)\, dz.$$

All of this mass experiences the same gravity $g$, so the force $f(y)$ and pressure $p(y)$ exerted from above on a place of area $A$ at altitude $y$ are

$$f(y) = m(y)g = Ag \int_y^\infty \rho(z)\, dz \qquad p(y) = \frac{f(y)}{A} = g \int_y^\infty \rho(z)\, dz.$$

Since it is easier to work with equations containing derivatives than ones containing integrals, I will differentiate both sides.

$$p'(y) = -g\rho(y).$$

## 2 Pressure dependence on density

The pressure and density of an ideal gas are related by the ideal gas law,

$$pV = nRT,$$

where $V$ is the volume of gas, $n$ is the number of moles of gas (a measure of the number of molecules of gas present), $R$ is a universal constant, and $T$ is the absolute temperature (normally measured in Kelvin). The quantities $n$ and $V$ are related to density, but more information is needed to make the connection. If we had the mass $m$ of the gas, then its density would be $\rho = \frac{m}{V}$. What we really need to know how much mass the gas has. The molar mass provides this information. If the gas has molar mass $M$, then its mass $m$ and density $\rho$ will be

$$m = Mn \qquad \rho = \frac{m}{V} = \frac{Mn}{V}$$

Using this to eliminate $n$ from the ideal gas law,

$$p = \frac{nRT}{V} = \frac{RT}{M}\rho.$$

This provides the relationship between pressure and density that is needed.

## 3 Differential equation

Combining the two relationships between pressure and density yields the differential equation

$$p'(y) = -g\rho(y) = -\frac{Mg}{RT}p(y).$$

Noting that $M$, $g$, $R$, and $T$ are assumed constant, this is just the equation for exponential decay. Its solution is

$$p(y) = p_0 e^{-\frac{Mg}{RT}y},$$

where $p_0 = p(0)$ is the pressure at sea level. The pressure (and density) decay exponentially with elevation $y$.