Math 142-2, Midterm

Solutions

Problem 1

Consider a damped spring given by the equation $mx'' + cx'|x'| + kx = 0$.

(a) Show that total energy can never increase. Can it decrease?

(b) Why is $c(x')^2$ not used for the damping term?

(c) What are the units of c?

(a) The energy is

$$\begin{aligned} E &= \frac{m}{2}(x')^2 + \frac{k}{2}x^2 \\ E' &= mx'x'' + kxx' \\ &= (mx'' + kx)x' \\ &= (-cx'|x'| - kx + kx)x' \\ &= -c|x'|^3 \\ &\leq 0 \end{aligned}$$

The energy cannot increase. It decreases whenever velocity is nonzero.

(b) If $c(x')^2$ were used instead, we would get $E' = -c(x')^3$, which leads to energy increase when velocity is negative. Damping terms should not lead to energy gain.

(c) Because of the addition,

$$\begin{aligned} [m][x''] &= [c][x']^2 \\ kg\, m\, s^{-2} &= [c] m^2\, s^{-2} \\ kg\, m^{-1} &= [c] \end{aligned}$$

Problem 1 (continued)

Consider a damped spring given by the equation $mx'' + cx'|x'| + kx = 0$.

(d) Determine using linearized stability analysis whether the system is stable, unstable, or neutrally stable.

(e) Is the system stable, unstable, or neutrally stable? Why?

(d) The equilibrium occurs when $x'' = 0$ and $x' = 0$, which implies $x = 0$. Thus, the equilibrium is at $x = 0$ and $v = 0$. Linearize about this a configuration.

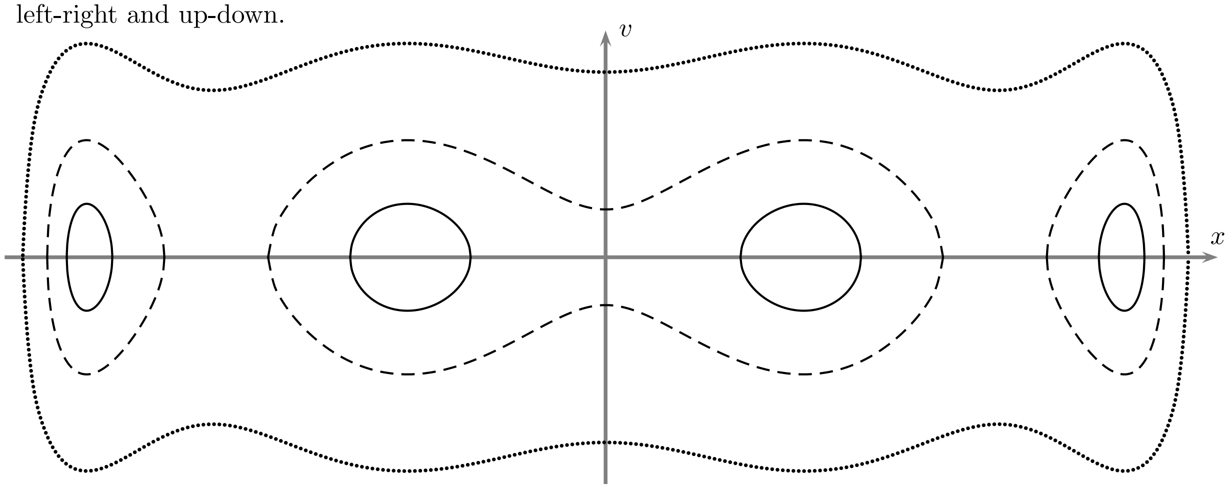
$$\begin{aligned} mx'' &= -cv|v| - kx = f(x, v) \\ mx'' &\approx f_x(0,0)x + f_v(0,0)v \\ &= -kx \end{aligned}$$

When linearized, the damping term vanishes, and the system is approximated by $mx'' + kx = 0$. Based on a linearized stability analysis, this system is neutrally stable, since a deviation from equilibrium will never grow with time, but it will also never decay back to equilibrium.

(e) The system is stable. If the system is not in equilibrium, it will be moving (at least most of the time), and we showed above that this causes energy loss. The energy loss causes the system to return (very slowly) to equilibrium, so the system is stable.

Problem 2

Consider the ODE $mx'' = f(x)$ for a particle, where the force $f(x)$ has the potential energy function $\phi(x)$. Below is part of the phase plane diagram for the resulting ODE. The phase plane is symmetrical left-right and up-down.



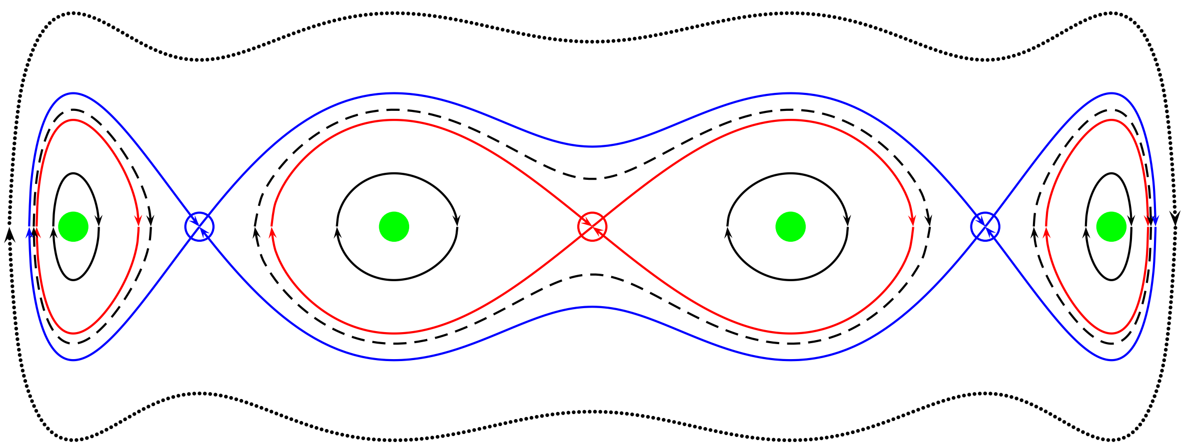
(a) The phase plane shows three energy levels: dotted, dashed, and solid. Which of these corresponds to the highest energy level? Which corresponds to the lowest energy level?

(b) On the phase plane diagram above, mark the stable equilibria with "•" and the unstable equilibria with "∘".

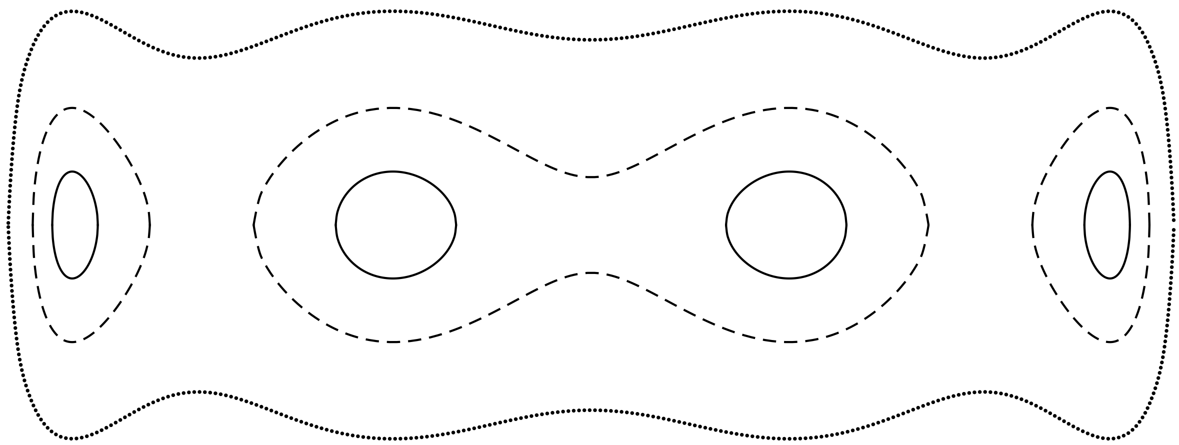
(c) On the phase plane diagram above, sketch the curves whose energy matches the energy of the unstable equilibria. These energy curves may contain more than one piece; be sure to sketch all of them.

(d) Put arrows on all of the curves (including the ones you drew in part (c)) to show the trajectories.

(a) The solid line has the lowest energy. The dotted line has the highest energy. Parts (b), (c), and (d) are shown on the phase plane below.

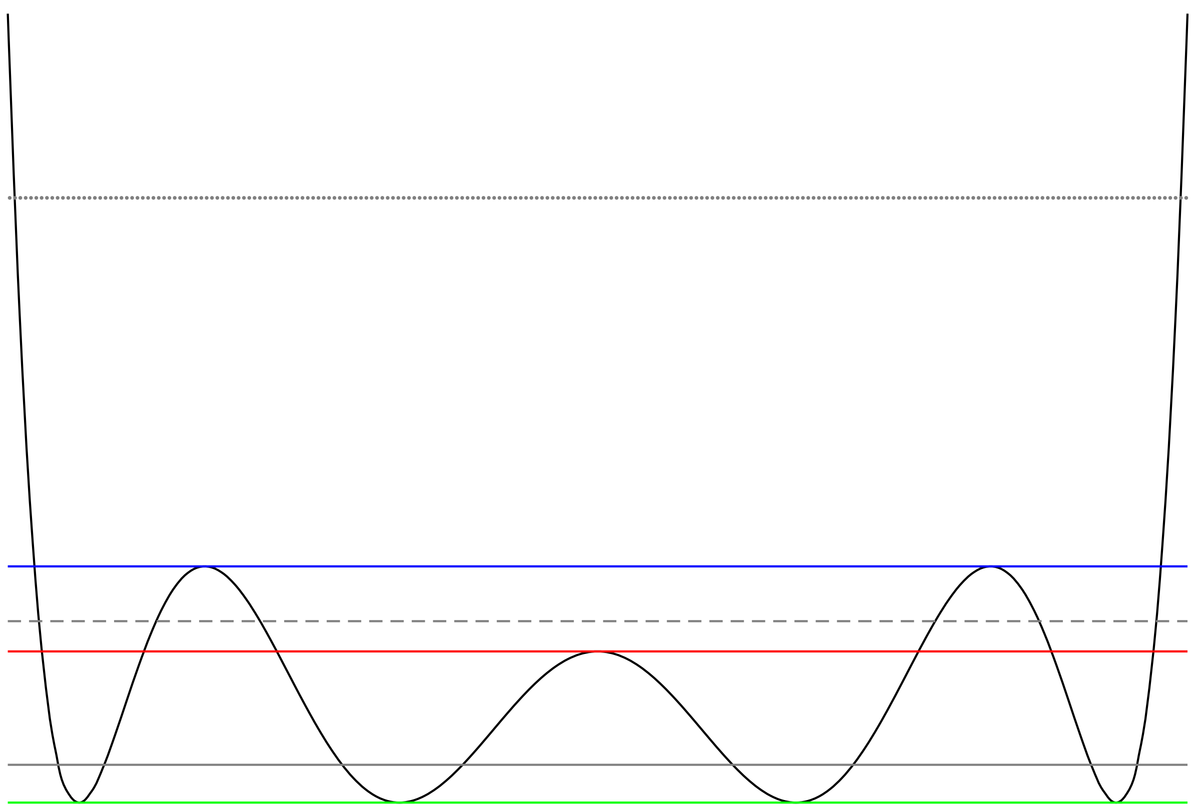


Problem 2 (continued)



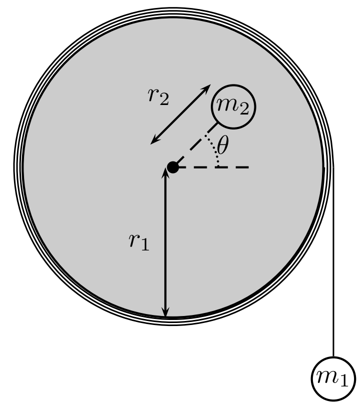
(e) Sketch the potential energy function. Show on your plot the energy levels corresponding to the three curves in the phase plane.

(e) The potential that created the phase plane above is plotted below, along with the three energy levels (in gray). The energy levels for the equilibria are also plotted below, though the question does not request them.



Problem 3

A pulley of radius r_1 has wrapped around it a long cable with an object of mass m_1 hanging from it. Another object of mass m_2 is attached to the pulley at a distance of r_2 from the pulley's center. Let θ be the polar angle the attached mass. Assume the cable is arbitrarily long.



(a) What is the potential energy of the system (in terms of θ)?

(b) What is the total energy of the system (in terms of θ and $\dot{\theta}$)?

(c) Show that this system obeys the ODE

$$(m_1 r_1^2 + m_2 r_2^2)\ddot{\theta} + r_2 m_2 g \cos\theta + r_1 m_1 g = 0.$$

(a) The height of m_2 is $r_2 \sin\theta$, so its potential energy is $r_2 m_2 g \sin\theta$. The height of m_1 is $y_0 + r_1\theta$, where y_0 is its height when $\theta = 0$. The total potential energy is then $\phi = r_2 m_2 g \sin\theta + (y_0 + r_1\theta) m_1 g$. Since a constant shift in potential energy does not matter, we can write

$$\phi = r_2 m_2 g \sin\theta + r_1 m_1 g \theta.$$

(b) The speed of m_1 is $r_1\dot{\theta}$, so its kinetic energy is $\frac{1}{2} m_1 r_1^2 \dot{\theta}^2$. The speed of m_2 is $r_2\dot{\theta}$, so its kinetic energy is $\frac{1}{2} m_2 r_2^2 \dot{\theta}^2$. The total energy is the sum of kinetic and potential energy, so

$$E = \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2)\dot{\theta}^2 + r_2 m_2 g \sin\theta + r_1 m_1 g \theta$$

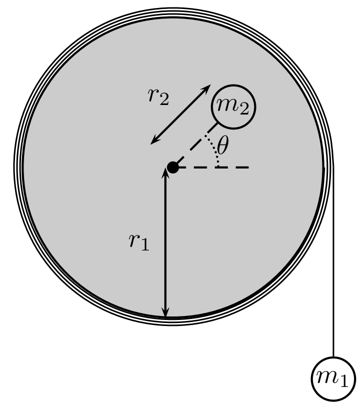
(c) From $\dot{E} = 0$,

$$\begin{aligned}
0 &= \dot{E} \\
&= (m_1 r_1^2 + m_2 r_2^2)\dot{\theta}\ddot{\theta} + r_2 m_2 g \dot{\theta} \cos\theta + r_1 m_1 g \dot{\theta} \\
&= [(m_1 r_1^2 + m_2 r_2^2)\ddot{\theta} + r_2 m_2 g \cos\theta + r_1 m_1 g]\dot{\theta} \\
0 &= (m_1 r_1^2 + m_2 r_2^2)\ddot{\theta} + r_2 m_2 g \cos\theta + r_1 m_1 g \\
(m_1 r_1^2 + m_2 r_2^2)\ddot{\theta} &= -r_2 m_2 g \cos\theta - r_1 m_1 g \\
\ddot{\theta} &= -\frac{r_2 m_2 g \cos\theta + r_1 m_1 g}{m_1 r_1^2 + m_2 r_2^2}
\end{aligned}$$

Problem 3 (continued)

(d) If $m_2 < M_e$, for some critical mass M_e, then this system has no equilibria. Find M_e.

(e) If $m_2 < M_e$, describe qualitatively the dynamical behavior of the system.



(d) To have equilibria, we need critical points in ϕ.

$$\begin{aligned} 0 &= \phi' \\ &= r_2 m_2 g \cos\theta + r_1 m_1 g \\ r_2 m_2 &\geq r_1 m_1 \\ m_2 &\geq \frac{r_1 m_1}{r_2} \\ &= M_e \end{aligned}$$

(e) The rotation rate will increase over time without bound, though it will fluctuate as it does so.