Math 142-1, Midterm

Solutions

Problem 1

Many physical formulas (e.g., $E = mc^2$, $E = \frac{hc}{\lambda}$, $V_T = \frac{kT}{q}$) contain fundamental constants (e.g., speed of light c, Plank's constant h, and Boltzmann's constant k), which is often seen as somewhat inconvenient. It was observed that since the choice of units is arbitrary, one could choose "more convenient" units. For example, if years are chosen for units of time and light-years (the distance light travels in a year) as the units for length, then $c = 1$. In fact, one can choose units so that $c = 1$, $h = 1$, and $k = 1$. The formulas above then "simplify" to $E = m$, $E = \frac{1}{\lambda}$, $V_T = \frac{T}{q}$. Omitting these constants from the formula turns out to be okay because they can always be unambiguously re-inserted later. For example, E has units of energy ($kg\,m^2\,s^{-2}$) but m has units of mass (kg). This can be balanced out with two factors of c (units of velocity, $m\,s^{-1}$) to recover the original equation $E = mc^2$. For each of the following "simplified" equations, determine what the originial equation was. A table listing units for all quantities is at right.

name	units
c	$m\,s^{-1}$
h	$kg\,m^2\,s^{-1}$
k	$kg\,m^2\,s^{-2}\,K^{-1}$
E	$kg\,m^2\,s^{-2}$
I	$kg\,s^{-2}$
λ	m
m	kg
ν	s^{-1}
p	$kg\,m\,s^{-1}$
q	C
T	K
V_T	$kg\,m^2\,s^{-2}\,C^{-1}$

(a) $E^2 = m^2 + p^2$

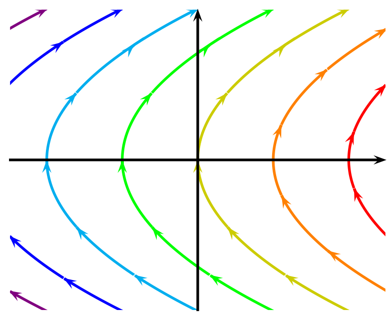
(b) $E = \nu$

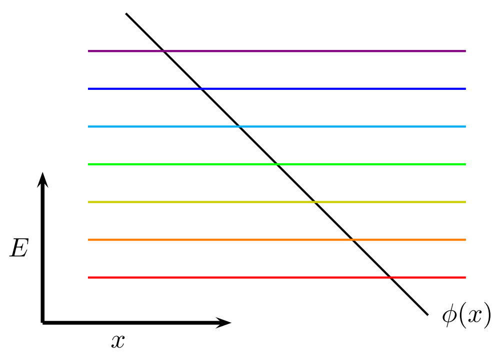
(c) $I = \dfrac{2\nu^3}{e^{\frac{\nu}{T}} - 1}$

(a) $E^2 = m^2c^4 + p^2c^2$
(b) $E = h\nu$
(c) $I = \dfrac{2h\nu^3}{c^2}\dfrac{1}{e^{\frac{h\nu}{kT}} - 1}$

Problem 2

Several energy levels for a system are shown in the phase plane below. (a) Mark all stable ("•") and unstable ("∘") equilibria. (b) Sketch energy contours corresponding to all unstable equilibria (energy contours may contain more than one component; be sure to sketch them all). (c) Add arrows to all contours, including the ones you have added. (d) Sketch the potential energy function and show the energy levels corresponding to the seven colored energy contours. (e) Construct a physical system that would exhibit motion consistent with the phase plane below.



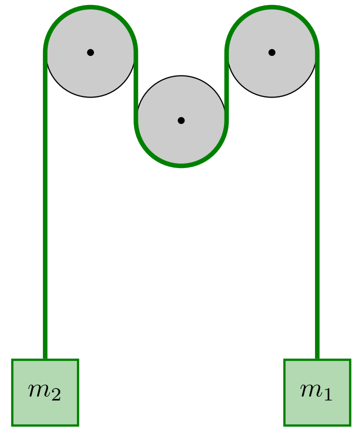


The curves also look similar to parabolas, so I will assume that they are to make things easier. The equation for the parabolas is then $x = a + bv^2$, with $b > 0$ fixed and a depends on the energy curve. This should be my energy equation, so lets make it look like one by making the v^2 term look like KE. $-\frac{m}{2b}x + \frac{1}{2}mv^2 = -\frac{am}{2b}$. The potential energy should be $-\frac{m}{2b}x$, with the right hand side being the constant total energy, which depends on a as desired. That is, potential energy should be proportional to x, with a negative sign. For example, $\phi(x) = -x$. Thus the potential energy function is a straight line with negative slope. Potential energy decreases from left to right and are evenly spaced. This allows us to complete part (d). Since our potential has no critical points, there are no equilibria. Thus, nothing should be done for parts (a) and (b).

(e) There are many possibilities for a physical system. For example, an object experiencing *negative* gravity. Alternatively, x might be defined to be the depth of some object (rather than the height of some object). In this case, gravity would act in the direction of increasing x, which is consistent with the phase plane. If $m_2 > m_1$, then the physical system in Problem 4 is consistent with this phase plane.

Problem 3

A massless rope is wrapped around three identical pulleys of radius r as shown in the figure. A mass is attached to each end of the rope. The pulleys are located at $(-2r, 0)$, $(0, -\frac{3}{2}r)$, and $(2r, 0)$. When both masses are at the same height, they are located at $(-3r, -s)$ and $(3r, -s)$. The location of mass m_1 at any time is $(3r, x(t) - s)$.



(a) What is the length ℓ of the rope?

(b) What is the total gravitational potential energy (ϕ)?

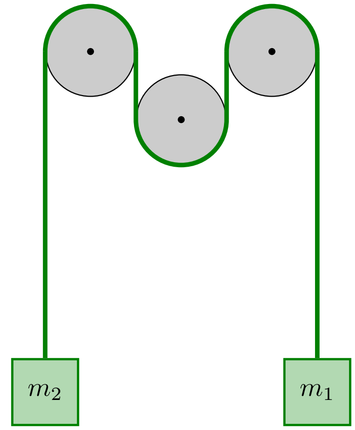
(c) The motion of the rope can be described by $\dot{x}(t)$. What is the total kinetic energy energy (KE)?

(a) The rope has two long vertical pieces (length s), two short ones (length $\frac{3}{2}r$), and three curved pieces (length πr). Thus, $\ell = 2s + 3r + 3\pi r$.

(b) If mass m_1 is at $(3r, x - s)$ then mass m_2 must be at $(3r, -x - s)$. $PE = m_1 g(x - s) + m_2 g(-x - s)$.

(c) Both masses move with the same speed (opposite directions). Thus, $KE = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$.

Problem 3 (continued)



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(d) What is the total energy (E)?

(e) Derive a second order ODE that describes the motion of the system.

(f) Find the equilibria of the system $(m_1 \neq m_2)$. Are they stable?

(g) Find the equilibria of the system $(m_1 = m_2)$. Are they stable?

(d) The total energy is

$$E = KE + \phi = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + m_1 g(x - s) + m_2 g(-x - s)$$

(e) An ODE is obtained from conservation of energy

$$\begin{aligned} 0 &= \dot{E} \\ &= (m_1 + m_2)\dot{x}\ddot{x} + (m_1 - m_2)g\dot{x} \\ 0 &= (m_1 + m_2)\ddot{x} + (m_1 - m_2)g \end{aligned}$$

(f), (g) Equilibria occur when $\dot{x} = \ddot{x} = 0$, which is possible only if $m_1 = m_2$. If the masses are unequal, there is no equilibrium. If the masses are equal, all configurations are equilibrium configurations. For (f), there are no equilibria to consider the stability of. For (g), stability is trickier. If I start an a nearby position at rest, I would stay there. But if I start at a nearby position with some velocity, I will go arbitrarily far from where I started. As long as these caveats are noted, I am okay with stable or unstable as answers, though unstable is a more defensible answer.