Math 142-1, Final

Solutions

Problem 1

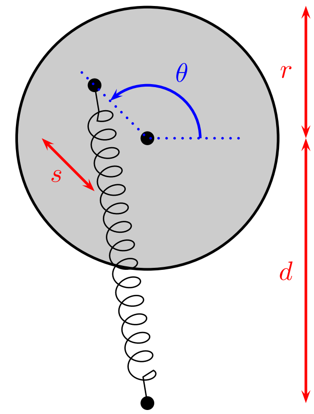
A 2D particle's location $\mathbf{x}$ obeys the equation of motion

$$m\ddot{\mathbf{x}} = -\frac{\mathbf{x}}{\|\mathbf{x}\|^2}.$$

Find the kinetic energy KE, potential energy ϕ, and total energy E of the system.

$$\begin{aligned}
\frac{d}{dt}(\mathbf{x}\cdot\mathbf{x}) &= 2\mathbf{x}\cdot\dot{\mathbf{x}} \\
\frac{d}{dt}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) &= 2\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}} \\
0 &= m\ddot{\mathbf{x}} + \frac{\mathbf{x}}{\|\mathbf{x}\|^2} \\
0 &= m\ddot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \frac{\mathbf{x}\cdot\dot{\mathbf{x}}}{\|\mathbf{x}\|^2} \\
0 &= \frac{1}{2}m\frac{d}{dt}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) + \frac{1}{2}\frac{1}{\mathbf{x}\cdot\mathbf{x}}\frac{d}{dt}(\mathbf{x}\cdot\mathbf{x}) \\
E &= \frac{1}{2}m\|\dot{\mathbf{x}}\|^2 + \ln\|\mathbf{x}\| \\
KE &= \frac{1}{2}m\|\dot{\mathbf{x}}\|^2 \\
\phi &= \ln\|\mathbf{x}\|
\end{aligned}$$

Problem 2



A wheel with total mass m (evenly distributed throughout) and radius r spins freely and without friction about its center. One end of a spring is attached to the wheel at a distance of s from the center of the wheel. The other end of the spring is fixed to a point d below the wheel's center. Find the equations of motion for the wheel, parameterized by the polar angle θ of the spring's attachment point. Your answer should be an ODE of the general form $\ddot{\theta} = f(\theta, \dot{\theta}, t)$. Hint: formulating energy first is easier.

I will start with kinetic energy. A chunk of the wheel at location $\mathbf{x} = \begin{pmatrix} R\cos(\phi+\theta) \\ R\sin(\phi+\theta) \end{pmatrix}$ will have velocity $\mathbf{v} = R\theta' \begin{pmatrix} -\sin(\phi+\theta) \\ \cos(\phi+\theta) \end{pmatrix}$. The chunk has mass $\rho\Delta A$, where $\rho = \frac{m}{\pi r^2}$. Then, $KE = \frac{1}{2}m\|\mathbf{v}\|^2 = \frac{m\Delta A}{2\pi r^2}R^2(\theta')^2$. In polar, $\Delta A = R\,d\phi\,dR$.

$$\begin{aligned} KE &= \int_0^r \int_0^{2\pi} \left(\frac{m}{2\pi r^2}R^2(\theta')^2\right) R\,d\phi\,dR \\ &= \left(\int_0^r R^3\,dR\right)\left(\int_0^{2\pi} d\phi\right)\left(\frac{m}{2\pi r^2}(\theta')^2\right) \\ &= \frac{r^4}{4}(2\pi)\left(\frac{m}{2\pi r^2}(\theta')^2\right) \\ &= \frac{mr^2}{4}(\theta')^2 \end{aligned}$$

Next, I need potential energy.

$$\begin{aligned} L &= \sqrt{(s\cos\theta)^2 + (s\sin\theta + d)^2} = \sqrt{s^2+d^2+2sd\sin\theta} \\ PE &= \frac{1}{2}k(L-\ell)^2 = \frac{1}{2}k\left(\sqrt{s^2+d^2+2sd\sin\theta} - \ell\right)^2 \end{aligned}$$

I can not assemble these to get the energy and use conservation of energy to get the ODE.

$$\begin{aligned} E &= \frac{mr^2}{4}(\theta')^2 + \frac{1}{2}k\left(\sqrt{s^2+d^2+2sd\sin\theta} - \ell\right)^2 \\ 0 &= \dot{E} \\ &= \frac{mr^2}{2}\theta'\theta'' + k\frac{\left(\sqrt{s^2+d^2+2sd\sin\theta} - \ell\right)}{2\sqrt{s^2+d^2+2sd\sin\theta}}(2sd\cos\theta)\theta' \\ &= \frac{mr^2}{2}\theta'\theta'' + ksd\cos\theta\left(1 - \frac{\ell}{\sqrt{s^2+d^2+2sd\sin\theta}}\right)\theta' \\ 0 &= \frac{mr^2}{2}\theta'' + ksd\cos\theta\left(1 - \frac{\ell}{\sqrt{s^2+d^2+2sd\sin\theta}}\right) \\ \theta'' &= \frac{2ksd}{mr^2}\cos\theta\left(\frac{\ell}{\sqrt{s^2+d^2+2sd\sin\theta}} - 1\right) \end{aligned}$$

Problem 3

Solve the PDE for $f(x,y,t)$, subject to $f(x,y,0)=g(x,y)$:

$$f_t(x,y,t) - yf_x(x,y,t) + xf_y(x,y,t) = 1.$$

This is a method of characteristics problem. Let $x(t)$ and $y(t)$ be the trajectory of a characteristic.

$$\begin{aligned}
\frac{d}{dt}f(x(t),y(t),t) &= f_t(x,y,t) + x'f_x(x,y,t) + y'f_y(x,y,t) \\
&= f_t(x,y,t) - yf_x(x,y,t) + xf_y(x,y,t) \qquad \text{where } x'=-y \text{ and } y'=x \\
&= 1
\end{aligned}$$

Next, solve for x and y subject to $x(0)=x_0$ and $y(0)=y_0$.

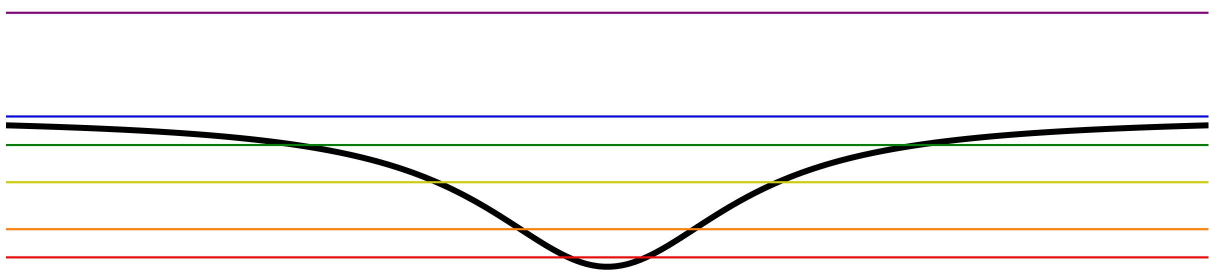
$$\begin{aligned}
x' &= -y \\
y' &= x \\
x'' &= -y' = -x \\
x &= c_0\cos t + c_1\sin t \\
x(0) &= c_0 = x_0 \\
y &= -x' \\
&= c_0\sin t - c_1\cos t \\
y(0) &= -c_1 = y_0 \\
x &= x_0\cos t - y_0\sin t \\
y &= x_0\sin t + y_0\cos t \\
x\cos t + y\sin t &= x_0\cos t\cos t - y_0\sin t\cos t + x_0\sin t\sin t + y_0\cos t\sin t = x_0 \\
x\sin t - y\cos t &= x_0\cos t\sin t - y_0\sin t\sin t - x_0\sin t\cos t - y_0\cos t\cos t = -y_0 \\
x_0 &= x\cos t + y\sin t \\
y_0 &= -x\sin t + y\cos t \\
\frac{d}{dt}f(x(t),y(t),t) &= 1 \\
f(x(t),y(t),t) &= f(x(0),y(0),0) + t \\
&= g(x_0,y_0) + t \\
&= g(x\cos t + y\sin t, -x\sin t + y\cos t) + t
\end{aligned}$$

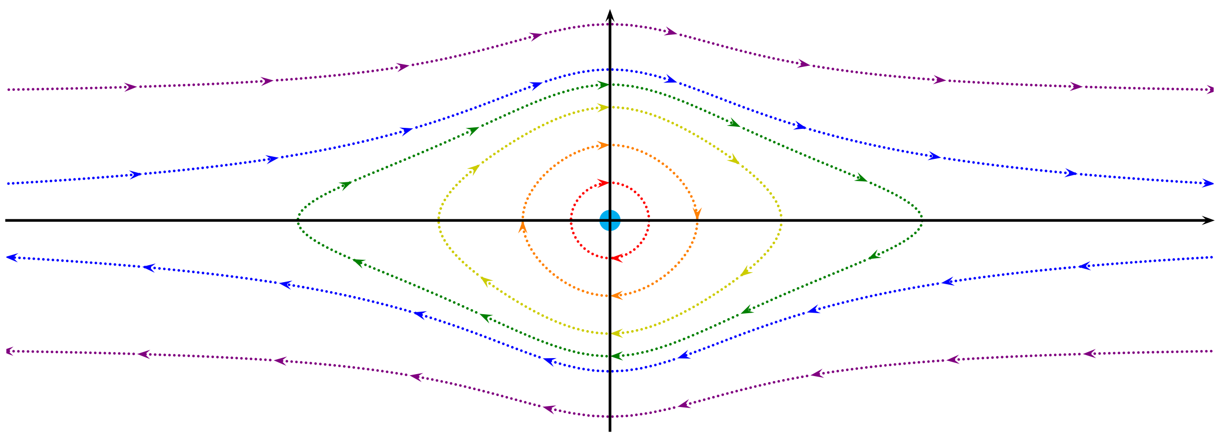
Problem 4

Sketch the phase plane for the ODE $\ddot{x} + x(x^2+1)^{-2} = 0$. Your sketch should include representative trajectories with arrows, including trajectories through unstable equilibria (if any). Mark all stable ("•") and unstable ("◦") equilibria.

$$0 = \ddot{x} + \frac{x}{(x^2+1)^2}$$
$$0 = \ddot{x}\dot{x} + \frac{x\dot{x}}{(x^2+1)^2}$$
$$E = \frac{1}{2}\dot{x}^2 - \frac{1}{2(x^2+1)}$$

Plotting the energy first will help with the phase plane.

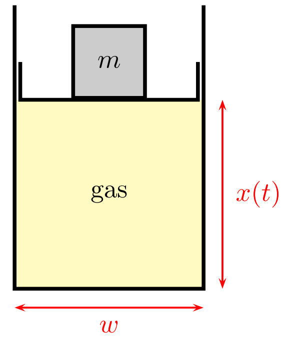




Equilibria can be found by letting $\dot{x} = \ddot{x} = 0$ in the equation of motion, which produces only the solution $x = 0$. Since the energy has a minimum here, this equilibrium is stable.

Problem 5

An idea gas is contained within a box with a moveable top. This wall is able to move up or down. Assume the walls are well-insulated. Assume that a constant fraction α of the total energy of the gas is translational energy. A mass m sits on top of the moveable ceiling, allowing it to move up and down by compressing the gas under the force of gravity, confining the gas to a $w \times w \times x(t)$ volume. Let $x(t) = x_0$ be the height of the gas volume when the mass experiences no net force. Find the equations of motion for the block. You do not need to solve them. (Hint: this related to one of the problems you did in groupwork.)

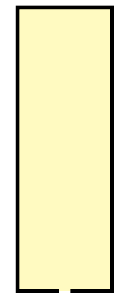


We need a relationship between the height of the ceiling and the force that the gas applies on the ceiling. We saw in groupwork that $PV^\gamma = \text{const}$, with $\gamma = 1 + \frac{2}{3}\alpha$, which provides the necessary relationship.

$$\begin{aligned}
A &= w^2 \\
c_1 &= PV^\gamma \\
&= \frac{F}{A}(Ax)^\gamma \\
F &= c_1 A (Ax)^{-\gamma} \\
m\ddot{x} &= c_1 w^2 (w^2 x)^{-\gamma} - mg \\
m\ddot{x} &= c_1 w^{2-2\gamma} x^{-\gamma} - mg \\
m\ddot{x} &= c_2 x^{-\gamma} - mg \\
0 &= c_2 x_0^{-\gamma} - mg \\
c_2 &= mg x_0^\gamma \\
m\ddot{x} &= mg\left(\left(\frac{x_0}{x}\right)^\gamma - 1\right) \\
\ddot{x} &= g\left(\left(\frac{x_0}{x}\right)^\gamma - 1\right)
\end{aligned}$$

Problem 6

A light bottle is filled with gas under high pressure. A plug on the bottom is removed, opening up a hole for gas to escape. When this occurs, the bottle flies into the air like a rocket. Explain how this result would be predicted using our gas model. The stationary bottle has no momentum or energy, but after the plug is removed and the bottle goes flying, it does. You should explain where the *momentum* and *energy* come from and how they get there. You do not need to do any detailed calculations. (A simple argument like "the gas goes down, so the bottle must go up to conserve momentum" would should show that the bottle must rise, but it does not tell us *why* or *how* this occurs. The goal of this exercise is to work out what is going on at the level of the particles to provide an explanation not only for what must occur but also why and how.) Be sure your explanation correctly accounts for these observations: (1) this experiment will also work in a vacuum (2) the bottle will not go flying if the plug is not removed and (3) the temperature and energy of a gas in a uniformly translating sealed bottle does not change over time.



When particles bounce off the top, they push the bottle up. When they bounce off the bottom, they push the bottle back down. In the absence of a hole, these will cancel each other out. When there is a hole, some of the gas particles that would normally have bounced off the plug instead escape through the hole. As a result, more momentum is deposited at the top wall than at the bottom, causing the bottle to get upward momentum. (Momentum is conserved; the momentum of the gas rushing out the hole is equal to the upward momentum of the bottle and the gas it contains.)

In fact, the situation is actually slightly more complicated than this. (Indeed, if the above explanation was all there was to it, the bottle would be unable to gain energy.) The reason for this is related to why a moving wall can cause gas to gain or lose energy. A particle bouncing off a wall that is moving away will be moving slower when it recoils. A particle bouncing off a wall that is moving towards it will be moving faster afterwards. As the bottle moves up, it causes particles that are moving upwards to be going a little faster and the particles moving downwards to be moving a bit slower. On its own, this effect actually causes the gas particles to have a little more energy than they would have if the bottle was not moving (if I want to make a $10\,kg$ gas container move, I have to give it kinetic energy. If that container is holding $0.1\,kg$ of gas, I have to give that gas kinetic energy, too.)

The missing piece is that the particles that escape the hole are the particles that are moving slower. Particles normally deposit energy at the top wall and get it back at the bottom wall. Particles that escape deposit their energy at the top wall but never get it back. That energy is used to accelerate the bottle and the other gas in the bottle. Since the gas that escapes has less kinetic energy, it is colder than it was originally. This is the source of the energy.

Problem 7

An initially piecewise constant density profile leads to two shocks and no rarefactions. Show that the shocks must eventually merge and the resulting shock moves with a velocity that is between the velocities of the original shocks. You may assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$. (If you are able to solve this problem without assuming any particular traffic following model, you will get extra credit equal in value to one problem on this exam.)

Two shocks and no rarefactions implies three regions of constant density in increasing order. Let them be $\rho_0 < \rho_1 < \rho_2$. Let $u_0 = u(\rho_0)$, $u_1 = u(\rho_1)$, and $u_2 = u(\rho_2)$. Then, $u_0 > u_1 > u_2$.

Recall that a shock's velocity is between that of the characteristic velocities $q'(\rho)$ of the densities on either side. To see this, note that by the mean value theorem we have

$$q'(\rho_s) = \frac{q(\rho_0) - q(\rho_1)}{\rho_0 - \rho_1} = s_{01}$$

for some $\rho_0 < \rho_s < \rho_1$. We required $q'(\rho)$ to be decreasing, from which it follows that $q'(\rho_0) > q'(\rho_s) = s_{01} > q'(\rho_1)$.

From $q'(\rho_0) > s_{01} > q'(\rho_1) > s_{12} > q'(\rho_2)$, we have $s_{01} > s_{12}$, which implies that the left shock moves faster than the right shock and will therefore eventually catch up with it.

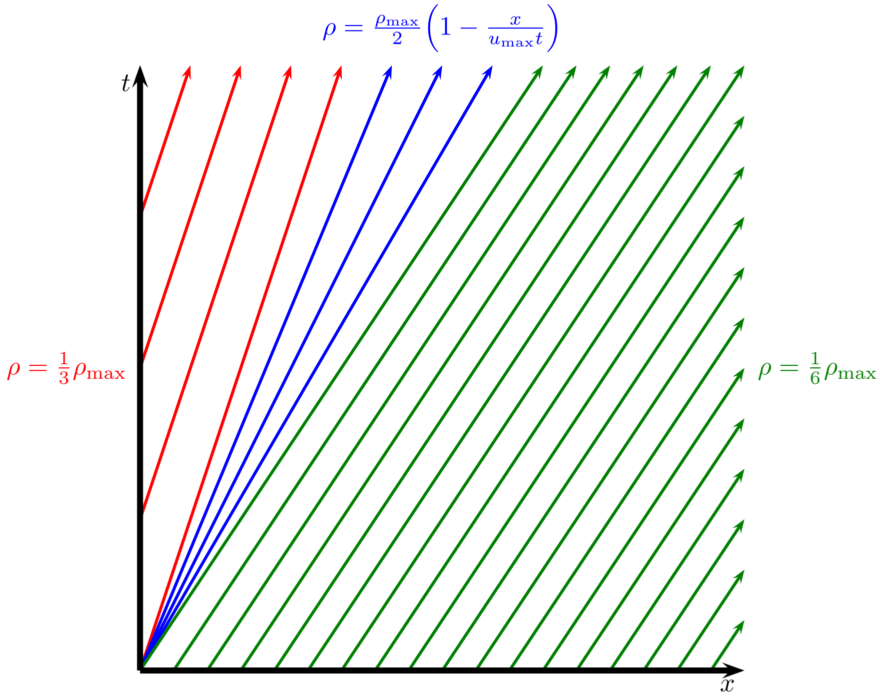
The speeds of the shocks are:

$$\begin{aligned}
s_{01} &= \frac{q(\rho_0) - q(\rho_1)}{\rho_0 - \rho_1} \\
s_{12} &= \frac{q(\rho_1) - q(\rho_2)}{\rho_1 - \rho_2} \\
s_{02} &= \frac{q(\rho_0) - q(\rho_2)}{\rho_0 - \rho_2} \\
A &= \rho_1 - \rho_0 \\
B &= \rho_2 - \rho_1 \\
(A + B)s_{02} &= As_{01} + Bs_{12} \\
s_{02} &= \frac{A}{A+B}s_{01} + \frac{B}{A+B}s_{12} \\
\lambda &= \frac{A}{A+B} \qquad 0 < \lambda < 1 \\
s_{02} &= \lambda s_{01} + (1 - \lambda)s_{12}
\end{aligned}$$

We can see that s_{02} is interpolated between s_{01} and s_{12} and is thus between them.

Problem 8

Determine the density of traffic on a semi-infinite road ($x \geq 0$) for all future times subject to the initial density profile $\rho(x,0) = \frac{1}{6}\rho_{\max}$ (for $x \geq 0$) and boundary conditions $q(0,t) = \frac{2}{9}\rho_{\max}u_{\max}$ (for $t \geq 0$). Assume $\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$.



$$\hat{u}(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$
$$\hat{q}(\rho) = u_{\max}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)$$
$$\hat{q}'(\rho) = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$$

The initial conditions have characteristics (green) of velocity $q'\left(\frac{1}{6}\rho_{\max}\right) = \frac{2}{3}u_{\max}$.
Since the boundary condition is

$$q = \frac{2}{9}\rho_{\max}u_{\max} = u_{\max}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)$$
$$\frac{\rho}{\rho_{\max}}\left(1 - \frac{\rho}{\rho_{\max}}\right) - \frac{2}{9} = 0$$
$$-\left(\frac{\rho}{\rho_{\max}}\right)^2 + \frac{\rho}{\rho_{\max}} - \frac{2}{9} = 0$$
$$-\left(\frac{\rho}{\rho_{\max}} - \frac{1}{3}\right)\left(\frac{\rho}{\rho_{\max}} - \frac{2}{3}\right) = 0$$

the density must look like $\rho = \frac{1}{3}\rho_{\max}$ or $\rho = \frac{2}{3}\rho_{\max}$. The characteristics (red) must be entering the domain (otherwise the boundary condition would not affect the answer!), which means the boundary condition must look like $\rho = \frac{1}{3}\rho_{\max}$, which corresponds to characteristic velocity $q'\left(\frac{1}{3}\rho_{\max}\right) = \frac{1}{3}u_{\max}$.

This leaves a gap (blue) with no characteristics, which means we have a rarefaction. Let the rarefaction density be $\rho(x,t)$. The characteristic speed is $\hat{q}'(\rho) = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$.

$$
\begin{aligned}
x &= \hat{q}'(\rho)t \\
&= u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)t \\
u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right) &= \frac{x}{t} \\
\frac{2\rho}{\rho_{\max}} &= 1 - \frac{x}{u_{\max}t} \\
\rho(x,t) &= \frac{\rho_{\max}}{2}\left(1 - \frac{x}{u_{\max}t}\right)
\end{aligned}
$$

Finally, we must put these together into the solution based on the region we are in.

$$
\rho(x,t) = \begin{cases} \frac{1}{3}\rho_{\max} & x < \frac{1}{3}u_{\max}t \\ \frac{\rho_{\max}}{2}\left(1 - \frac{x}{u_{\max}t}\right) & \frac{1}{3}u_{\max}t < x < \frac{2}{3}u_{\max}t \\ \frac{1}{6}\rho_{\max} & \frac{2}{3}u_{\max}t < x \end{cases}
$$

Problem 9

A physical system is observed to have the following properties:

- There is exactly one equilibrium (at $x = 0$); the equilibrium is unstable
- For sufficiently large $|x|$, the system exhibits decay (energy is lost)

Construct a model (an ODE) for a 1D system which has these properties. (Hint: no linear ODE satisfies both properties; the damping coeffient can depend on x.)

Consider a model of the form $\ddot{x} + c\dot{x} + kx = 0$. If $c > 0$ and $k > 0$, then this model predicts decay. That satisfies the second requirement, but it also has a stable equilibrium. On the other hand, if $c < 0$ and $k > 0$, the equilibrium will be unstable. But in this case, energy grows for large $|x|$, too. We really need c to depend on x. That suggests a model of the form $\ddot{x} + (ax^2 - b)\dot{x} + kx = 0$, which has the desired properties.

Problem 10

SI units are a system of units that were chosen more or less arbitrarily to be convenient for everyday use. An alternative approach to constructing units is to select units where fundamental constants of nature become one. To fix units for distance (m), time (s), mass (kg), charge (C), and temperature (K), one must choose five fundamental constants. Planck units are a system of units designed to be purely non-arbitrary. Planck units are obtained by setting the gravitational constant (G), the reduced Planck constant ($\hbar$), the speed of light (c), the Coulomb constant (k_e), and the Boltzmann constant (k) to be one. In these units, $E = mc^2$ can be simply written as $E = m$, and the entropy of a black hole is $S = \frac{A}{4}$, where A is its area. How must the relationship between the entropy of a black and its area be written if SI units are used instead?

Quantity	Units
c	$m\,s^{-1}$
G	$kg^{-1}\,m^3\,s^{-2}$
$\hbar$	$kg\,m^2\,s^{-1}$
k_e	$kg\,m^3\,s^{-2}\,C^{-2}$
k	$kg\,m^2\,s^{-2}\,K^{-1}$
E	$kg\,m^2\,s^{-2}$
m	kg
S	$kg\,m^2\,s^{-2}\,K^{-1}$
A	m^2

$$\begin{aligned}
S &= c^j G^n \hbar^p k_e^q k^r \frac{A}{4} \\
[S] &= [c^j][G^n][\hbar^p][k_e^q][k^r][A] \\
kg\,m^2\,s^{-2}\,K^{-1} &= (m\,s^{-1})^j (kg^{-1}\,m^3\,s^{-2})^n (kg\,m^2\,s^{-1})^p (kg\,m^3\,s^{-2}\,C^{-2})^q (kg\,m^2\,s^{-2}\,K^{-1})^r (m^2) \\
kg\,m^2\,s^{-2}\,K^{-1} &= kg^{-n+p+q+r}\, m^{j+3n+2p+3q+2r+2}\, s^{-j-2n-p-2q-2r}\, C^{-2q}\, K^{-r} \\
1 &= -n+p+q+r \\
2 &= j+3n+2p+3q+2r+2 \\
-2 &= -j-2n-p-2q-2r \\
0 &= -2q \\
-1 &= -r
\end{aligned}$$

Immediately, $q = 0$ and $r = 1$. Eliminating n with the first equation we get $n = p$ and

$$\begin{aligned}
-2 &= j + 5p \\
0 &= -j - 3p
\end{aligned}$$

Adding these equations we get $-2 = 2p$, so $p = -1$ and $j = 3$. Substituting back we get $n = -1$.

$$S = \frac{c^3 k A}{4G\hbar}$$