Math 142-2, Homework 2

Your name here

Problem 10.4

Consider Coulomb friction, equation 10.3.

(a) What is the sign of γ?

Your solution goes here

(b) What is the dimension of γ?

Your solution goes here

(c) Assume that $dx/dt = 0$, then F_f could be any value such that $|F_f| \leq \gamma$. What values of x are then equilibrium positions of the spring-mass system?

Your solution goes here

Problem 10.5

Assume the same form of friction as in exercise 10.4. If initially $x = x_0 > 0$ and the velocity is $v_0 > 0$, then at what time does the mass of a spring-mass system first stop moving to the right? Will the mass continue to move after that time?

Your solution goes here

Problem 10.9

A particle not connected to a spring, moving in a straight line, is subject to a *retardation force* of magnitude $\beta(dx/dt)^n$, with $\beta > 0$.

(a) Show that if $0 < n < 1$, the particle will come to rest in a finite time. How far will the particle travel, and when will it stop?

Your solution goes here

(b) What happens if $n = 1$?

Your solution goes here

(c) What happens if $1 < n < 2$?

Your solution goes here

(d) What happens if $n > 2$?

Your solution goes here

Problem 12.6

Show that the local maximum or minimum for the displacement of an underdamped oscillation does not occur halfway between the times at which the mass passes through its equilibrium position. However, show that the time period between successive local maxima (or minima) is constant. What is this period?

Your solution goes here

Problem 12.7

Consider a spring-mass system with linear friction (but without gravity). Suppose that there is an additional force, $Bcos\omega_0 t$, a periodic forcing function. Assume that B and ω_0 are known.

(a) If the coefficient of friction is zero (i.e., $c = 0$), then determine the general solution of the differential equation. Show that the solution is oscillatory if $\omega_0 \neq \sqrt{k/m}$. Show that the solution algebraically grows in time if $\omega_0 = \sqrt{k/m}$ (this is called resonance and occurs if the forcing frequency ω_0 is the same as the natural frequency $\sqrt{k/m}$).

Your solution goes here

(b) If $c^2 < 4mk$, show that the general solution consists of the sum of oscillatory terms (of frequency ω_0) and terms which exponentially decay in time. Thus for sufficiently large times the motion is accurately approximated by an oscillation of constant amplitude. What is that amplitude? Note it is independent of initial conditions.

Your solution goes here

(c) If $c^2 \ll 4mk$, show that for large times the oscillation approximately obeys the following statements: If the forcing frequency is less than the natural frequency ($\omega_0 < \sqrt{k/m}$), then the mass oscillates "in phase" with the forcing function (i.e., when the forcing function is a maximum, the stretching of the spring is a maximum, and vice versa). On the other hand, if the forcing frequency is greater than the natural frequency ($\omega_0 > \sqrt{k/m}$), then the mass oscillates "180 degrees out of phase" with the forcing function (i.e., when the forcing function is a maximum, the compression of the spring is a maximum and vice versa).

Your solution goes here

(d) In part (b) the ratio of the amplitude of oscillation of the mass to the amplitude of the forcing function is called the response. If $c^2 \ll 4mk$, then at what frequency is the response largest?

Your solution goes here

Problem 13.1

Assume that friction is sufficiently large ($c^2 > 4mk$).

(a) Show that the mass either decays to its equilibrium position (without passing through it), or that the mass shoots past its equilibrium position exactly once before returning monotonically towards its equilibrium position.

Your solution goes here

(b) If the initial position x_0 of the mass is positive, then show that the mass crosses its equilibrium position only if the initial velocity is sufficiently negative. What is the value of this critical velocity? Does it depend in a reasonable way on x_0, c, m, and k?

Your solution goes here

Problem 14.2

Consider a mass m located at $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$, where x and y are unknown functions of time. The mass is free to move in the $x - y$ plane without gravity (i.e., it is not connected to the origin via the shaft of a pendulum) and hence the distance L from the origin may vary with time.

(a) Using polar coordinates as introduced in Sec. 14, what is the velocity vector?

Your solution goes here

(b) Show that the acceleration vector $\mathbf{a} = (d^2/dt^2)\mathbf{x}$ is

$$\mathbf{a} = \left(L\frac{d^2\theta}{dt^2} + 2\frac{dL}{dt}\frac{d\theta}{dt}\right)\boldsymbol{\theta} + \left(\frac{d^2L}{dt^2} - L\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{r}$$

Your solution goes here

Problem 14.3

Consider a mass m located at $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ and only acted upon by a force in the direction of $\mathbf{r}$ with magnitude $-g(L)$ depending only on $L \equiv \|\mathbf{x}\|$, called a *central force*.

(a) Derive the differential equations governing the angle θ and the distance L [Hint: Use the equation for a determined in exercise 14.2b],

Your solution goes here

(b) Show that $L^2(d\theta/dt)$ is constant [Hint: Differentiate $L^2(d\theta/dt)$ with respect to t]. This law is called *Conservation of Angular Momentum.*

Your solution goes here

Additional Problem

A rigid body can be modeled as a collection of point masses at positions $\mathbf{x}_p$ with masses m_p. Let $\mathbf{x}(t)$ the the position of some fixed point in the rigid body, and let $\mathbf{R}(t)$ be a rotation matrix ($\mathbf{R}\mathbf{R}^T = \mathbf{I}$, $\det(\mathbf{R}) = 1$). Relative to this point, we can say that $\mathbf{x}_p = \mathbf{x} + \mathbf{R}\mathbf{r}_p$, where $\mathbf{r}_p$ represent the displacements of the point masses from the reference point $\mathbf{x}$, when the rigid body is in its reference orientation ($\mathbf{R} = \mathbf{I}$). With this representation, $\mathbf{r}_p$ is constant for each point mass; only $\mathbf{x}(t)$ and $\mathbf{R}(t)$ depend on time.

(a) Show that $\dot{\mathbf{R}}\mathbf{R}^T$ is skew-symmetric (that is, show that $\dot{\mathbf{R}}\mathbf{R}^T = -(\dot{\mathbf{R}}\mathbf{R}^T)^T$). Because of this, we can define the angular velocity to be $\boldsymbol{\omega}$, where

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad \boldsymbol{\omega}^* = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \boldsymbol{\omega}^* = \dot{\mathbf{R}}\mathbf{R}^T$$

and $\boldsymbol{\omega}^*$ is defined so that $\boldsymbol{\omega}^*\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}$. Why are we mathematically justified in defining $\boldsymbol{\omega}$ in this way? (It suffices to show that $\boldsymbol{\omega}$ can be computed for any $\mathbf{R}$ and $\dot{\mathbf{R}}$.) Note that this also provides the ODE for orientation, $\dot{\mathbf{R}} = \boldsymbol{\omega}^*\mathbf{R}$.

Your solution goes here

(b) Express the velocity $\mathbf{v}_p$ of each particle p in terms of the rigid body velocity ($\mathbf{v} = \dot{\mathbf{x}}$), angular velocity ($\boldsymbol{\omega}$), position ($\mathbf{x}$), and orientation ($\mathbf{R}$).

Your solution goes here

(c) Express the total mass (m) and momentum ($\mathbf{p}$) of the rigid body in terms of fixed per-particle states (m_i and $\mathbf{r}_i$) and the rigid body state ($\mathbf{x}$, $\mathbf{v}$, $\mathbf{R}$, and $\boldsymbol{\omega}$).

Your solution goes here

(d) Up to this point, we have not chosen the point in the rigid body where $\mathbf{x}$ is computed in any particular way. Show that the total momentum can be written as $\mathbf{p} = m\mathbf{v}$ if we choose $\mathbf{x}$ to be the center of mass,

$$\mathbf{x} = \frac{1}{m}\sum_p m_p \mathbf{x}_p$$

Assume for the rest of this problem that $\mathbf{x}$ is at the center of mass.

Your solution goes here

(e) Show that the total kinetic energy of the rigid body has the form

$$KE = \frac{m}{2}\|\mathbf{v}\|^2 + \frac{1}{2}\boldsymbol{\omega}^T \mathbf{J}\boldsymbol{\omega}$$

where $\mathbf{J}$ is called the inertia tensor. Show that $\mathbf{J} = \mathbf{R}\mathbf{J}_0\mathbf{R}^T$, where $\mathbf{J}_0$ is the constant intertia tensor in the rest configuration ($\mathbf{J}_0 = \mathbf{J}$ when $\mathbf{R} = \mathbf{I}$.) Provide a formula for $\mathbf{J}_0$ in terms of m_p and $\mathbf{r}_p$. You may assume that $\mathbf{J}_0$ and $\mathbf{J}$ are invertible.

Your solution goes here

(f) Define the angular momentum of a rigid body to be

$$\mathbf{L} \equiv \sum_p (\mathbf{x}_p - \mathbf{x}) \times m_p \mathbf{v}_p$$

Show that $\mathbf{L} = \mathbf{J}\boldsymbol{\omega}$.

Your solution goes here

(g) Define the force $\mathbf{f}$ and torque $\boldsymbol{\tau}$ to be the change in momentum and angular momentum ($\mathbf{f} = \dot{\mathbf{p}}$ and $\boldsymbol{\tau} = \dot{\mathbf{L}}$). Show that in the absence of forces ($\mathbf{f} = \mathbf{0}$) or torques ($\boldsymbol{\tau} = \mathbf{0}$), mass, momentum, angular momentum, and kinetic energy are conserved. That is, $\dot{m} = 0$, $\dot{\mathbf{p}} = \mathbf{0}$, $\dot{\mathbf{L}} = \mathbf{0}$, and $\dot{KE} = 0$. Conclude that an isolated rigid body translates uniformly ($\dot{\mathbf{v}} = \mathbf{0}$), and its rotation evolves according to

$$\dot{\mathbf{R}} = (\mathbf{R}\mathbf{J}_0^{-1}\mathbf{R}^T\mathbf{L})^*\mathbf{R},$$

where $\mathbf{J}_0$ and $\mathbf{L}$ are constant.

Your solution goes here