Math 142-2, Homework 6

Your name here

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Problem 64.1

Consider the linear car-following model, equation 64.2, with a response time T (a delay).
(a) Ignore this part.
(b) Assume the lead driver's velocity varies periodically

$$v_0 = \mathbf{Re}(1 + f_0 e^{i\omega t}).$$

Also assume the n-th driver's velocity varies periodically

$$v_n = \mathbf{Re}(1 + f_n e^{i\omega t}),$$

where f_n measures the amplification or decay which occurs. Show that

$$f_n = \left(1 + \frac{i\omega}{\lambda} e^{i\omega T}\right)^{-n} f_0,$$

where $0 < f_0 < 1$. Note: f_n will be complex, f_0 is real. Unlike the problem as stated in the book, I am assuming $f_0 < 1$, since otherwise the model predicts that even the first car stops frequently. If $f_0 \approx 0$, then the first car drives quite smoothly, and the effects on later cars are more meaningful. This alteration does not make the problem more difficult.

Your solution goes here

(c) Show the magnitude of the amplification factor f_n decreases with n if

$$\frac{\sin \omega T}{\omega} < \frac{1}{2\lambda}.$$

Your solution goes here

(d) Show that the above inequality holds for all ω only if $\lambda T < \frac{1}{2}$.

Your solution goes here

(e) Conclude that if the product of the sensitivity and the time lag is greater than $\frac{1}{2}$, it is possible for following cars to drive much more erratically than the leader. In this case we say the model predicts instability if $\lambda T > \frac{1}{2}$ (i.e., with a sufficiently long time lag). (This conclusion can be reached more expeditiously through the use of Laplace transforms.)

Your solution goes here

Problem 65.1

Determine the solution of $\frac{\partial \rho}{\partial t} = (\sin x)\rho$ which satisfies $\rho(x, 0) = \cos x$.

Your solution goes here

Problem 67.3

Suppose initially ($t = 0$) that the traffic density is $\rho = \rho_0 + \epsilon \sin x$, where $|\epsilon| \ll \rho_0$. Determine $\rho(x, t)$.

Your solution goes here

Problem 67.5

Based on a linear analysis, would you say $\rho = \rho_0$, a constant, is a stable or unstable equilibrium solution of equation 66.1?

Your solution goes here

Problem 68.1

Explain why a density wave moves forward for light traffic. Consider both cases in which the traffic is getting heavier down the road and lighter.

Your solution goes here

Problem 71.1

Experiments in the Lincoln Tunnel (combined with the theoretical work discussed in exercise 63.7) suggest that the traffic flow is approximately

$$q(\rho) = a\rho[\ln(\rho_{\max}) - \ln(\rho)]$$

(where a and $\rho_{\max}$ are known constants). Suppose the initial density $\rho(x,0)$ varies linearly from bumper-to-bumper traffic (behind $x = -x_0$) to no traffic (ahead of $x = 0$) as sketched in Fig. 71-6. Two hours later, where does $\rho = \rho_{\max}/2$?

Your solution goes here

Problem 71.9

Show that $\rho = f(x - q'(\rho)t)$ satisfies equation 71.1 for any function f. Note that initially $\rho = f(x)$. Briefly explain how this solution was obtained.

Your solution goes here

Additional Problem

The equations that describe compressible flow in 1D are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$
$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0$$
$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(Eu + pu) = 0$$

There are four variables here that must be determined (ρ, u, E, and p), so one additional equation is required. This takes the form of an *equation of state*, which is some equation expressing p in terms of the other quantities. An example of such an equation of state is the ideal gas law $p = \rho e(\gamma - 1)$, where e is related to E and the temperature T by

$$E = \rho e + \frac{1}{2}\rho u^2 \qquad e = kT.$$

var	meaning
ρ	density
u	velocity
e	internal energy
E	total energy
p	pressure
γ	gas-specific constant
T	temperature
k	specific heat capacity

(a) Deduce the SI units (kg, m, s, K) of u, e, E, p, γ, and k from the equations above, given that $\rho \to kg\, m^{-1}$ and $T \to K$. Note that we are assuming 1D, so some of these units will not be what you may be used to seeing in 3D.

Your solution goes here

(b) A conservation law is said to be in conservation form when written as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0,$$

where U is a vector of conserved quantities and F is a vector of quantities called *fluxes*. Show that if a differential equation of this form holds for a vector of quantities U, then for each component of U there is some corresponding conserved quantity. Identify the conserved quantity for each of the three components of U. You may assume periodic boundary conditions. (That is, consider the system to live in some interval $[a, b]$, where $U(a, t) = U(b, t)$, $p(a, t) = p(b, t)$, $F(a, t) = F(b, t)$, etc.)

Your solution goes here

(c) Consider again (b) on the interval $[a, b]$, but this time do not assume periodicity. Provide a physical interpretation for the role that the fluxes F play.

Your solution goes here