Math 142-2, Homework 4

Your name here

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Problem 50.3

Reconsider exercise 50.2. In the phase plane, sketch the isocline(s) along which $dF/dS = 0$. Also sketch the isocline(s) along which $dF/dS = \infty$. Indicate arrows as in Fig. 50-3. Explain on your diagram where all possible equilibrium populations are. (If needed, sketch all possible cases.)

Your solution goes here

Problem 50.4

Reconsider the predator-prey model of exercise 50.2 ($b \neq 0$). Determine the constant coefficient linear system of differential equations which governs the small displacements from the equilibrium populations,

$$F = \frac{k}{\lambda}, S = \frac{a}{c} - \frac{bk}{c\lambda} > 0.$$

Do this in two ways:

1. Using the Taylor series for a function of two variables (equation 44.5).

Your solution goes here

2. Using perturbation methods as described in Sec. 50.

Your solution goes here

Show that the two are equivalent. Do not attempt to solve the resulting system. [Hint: See exercise 50.5 for the answer.]

Your solution goes here

Problem 50.5

Show that your answer to exercise 50.4 can be put in the form:

$$\frac{dF_1}{dt} = \frac{k}{\lambda}(-bF_1 - cS_1) \qquad\qquad \frac{dS_1}{dt} = \left(\frac{a}{c} - \frac{bk}{c\lambda}\right)\lambda F_1,$$

where F_1, and S_1, are the displacements from the equilibrium populations of the fish and sharks respectively. Eliminate F_1 (from the second equation) to derive a second-order constant coefficient differential equation for S_1. Analyze that equation and determine the conditions under which the equilibrium population is stable.

Your solution goes here

Problem 50.7

Refer to exercise 50.4. Suppose that the equilibrium number of prey without predators a/b is smaller than the prey necessary to sustain the predators k/λ. Sketch the solution in the phase plane. [Hint: Use the behavior of the phase plane in the neighborhood of all equilibrium populations.]

Your solution goes here

Problem 50.12

Consider the following two-species population growth model:

$$\frac{dF}{dt} = F(a - bF - cS) \qquad\qquad \frac{dS}{dt} = S(-k + \lambda F - \sigma S).$$

How does this model differ from equation 50.1. Without explicitly determining the equilibrium population, assume that one exists (with both species nonzero) and analyze its linear stability.

Your solution goes here

Problem 50.13

Reconsider exercise 50.12.

(a) Sketch the phase plane in the neighborhood of the equilibrium population in which both populations are nonzero.

Your solution goes here

(b) Sketch the phase plane in the neighborhood of $S = 0$, $F = a/b$. (What is the ecological significance of this population?)

Your solution goes here

(c) Sketch the phase plane in the neighborhood of $S = 0, F = 0$.

Your solution goes here

(d) Use the information gained from parts (a)-(c) to sketch the entire phase plane. Describe the predator-prey interaction. Sketch typical time dependence of predators and prey.

Your solution goes here

Problem 54.4

Consider the competing species model, equation 54.1. Suppose $b = d = 0$.

(a) What are all possible equilibrium solutions? Show that $x = c/\sigma$, $y = a/k$ is an equilibrium population.

Your solution goes here

(b) Show that the system of differential equations describing populations near the equilibrium ($x = c/\sigma$, $y = a/k$) is

$$\frac{dx_1}{dt} = -\alpha y_1 \qquad \frac{dy_1}{dt} = -\beta x_1,$$

where x_1 and y_1 are the displacements from equilibrium. What are α and β? Show that $\alpha > 0$, $\beta > 0$.

Your solution goes here

(c) By eliminating either x_1 or y_1, determine whether this equilibrium solution is stable or unstable.

Your solution goes here

(d) Sketch the phase plane solution of equation 54.1 with $b = d = 0$. [Hint: Are there any approximately straight line solutions in the neighborhood of the equilibrium solution $x = c/\sigma$, $y = a/k$?]

Your solution goes here

Problem 54.5

Formulate one system of differential equations describing all of the following interactions in some region between species x and species y:

(a) The nutrients for both species are limited.

(b) Both species compete with each other for the same nutrients.

(c) There is a migration (from somewhere else) of species x into the region of interest at the rate W per unit time.

Your solution goes here

Problem 54.11

Consider

$$\frac{dF}{dt} = F(2 - 2F + G) \qquad \frac{dG}{dt} = G(1 - G + F)$$

(a) Give a brief explanation of each species' ecological behavior. (Account for each term on the right-hand side of the above differential equations.)

Your solution goes here

(b) Determine all possible equilibrium populations.

Your solution goes here

(c) In the phase plane, draw the isoclines corresponding to the slope of the solution being 0 and ∞.

Your solution goes here

(d) Introduce arrows indicating the direction of trajectories (time changes) of this ecosystem. The qualitative time changes should be indicated everywhere ($F > 0$, $G > 0$) in the phase plane.

Your solution goes here

(e) From the phase plane in part (d), briefly explain which (if any) of the equilibrium populations is stable and which unstable. Do not do a linearized stability analysis.

Your solution goes here

Problem 54.12

Two species, A and B, are in competition and are the prey of a third species C. What differential equations describe this ecological system?

Your solution goes here

Additional Problem

Consider that some quantities $x(t)$ and $y(t)$ (valid for all reals) obey the differential equations

$$\frac{dx}{dt} = \cos(y) \qquad \frac{dy}{dt} = \sin(x).$$

(a) Identify all of the equilibrium configurations.

Your solution goes here

(b) Since x and y will be periodic with period 2π, we can concentrate on the region with $0 \leq x < 2\pi$ and $0 \leq y < 2\pi$. There are four equilibrium locations in this region. Compute the linearization of the differential equation at each equilibrium position.

Your solution goes here

(c) For each of the four equilibria from (b), identify the equilibrium's stability (unstable, stable, or weakly/neutrally stable) and shape (node, saddle point, spiral/loop).

Your solution goes here

(d) Use the fact that $\frac{dy}{dx}$ is separable to compute an energy for the system. Your solution $E(x, y)$ should take the form of a simple polynomial in terms of $\cos(x)$, $\sin(x)$, $\cos(y)$, and $\sin(y)$.

Your solution goes here

(e) Show that, when evaluated at equilibrium values, $E(x_e, y_e)$ takes on only three distinct values: $c_0 < c_1 < c_2$. Show that c_0 and c_2 are minima and maxima of E and that these are the only minima and maxima of E.

Your solution goes here

(f) Show that $E(x, y) = c_1$ along straight lines and identify these lines. Trajectories corresponding to $c_0 < c < c_2$ with $c \neq c_1$ are bounded by the c_1 lines and contain no equilibria and so must represent closed curves.

Your solution goes here

(g) Use the information gathered above to sketch the phase plane for the system under consideration. Include the c_1 lines and sample trajectories in your diagram.

Your solution goes here

(h) What do you suppose will happen to a system that starts on one of the c_1 lines? (Describe the shape of its trajectory and what happens in the limits $t \to \infty$ and $t \to -\infty$.)

Your solution goes here