Ideal Gas Law

Lecture Notes (Math 142-1)

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1 Gases as collection of particles

- Gases have several measurable properties
  - Density ρ
  - Pressure P
  - Temperature T
  - Composition (water vapor, nitrogen, oxygen, helium, carbon dioxide, etc.)
  - Mass M
  - Volume V
- Want a relationship between these
  - Some are obvious, such as $\rho = \frac{M}{V}$
- Assumptions
  - Container is a box
  - Gas has one type of substance
  - Gas is made of small descrete particles (molecules)
    * Identical
    * Spherical
    * Tiny size
  - Molecules do not interact
    * Sparse gas, low density
    * low probably of collisions

2 One dimension

- Start with 1D, will lift to 3D later
- Start with one particle

2.1 Microscopic

- One particle
  - Mass: m
  - velocity: v
- Container length: L
  - Particle collides with walls of the container
  - Elastic (no energy loss)
- Collision
  - Velocity before: v_0
  - Velocity after: v_1
  - No energy loss: $\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2$
  - Particle reverses direction
  - $v_1 = -v_0$
  - Change in particle momentum due to collision: $\Delta p = mv_1 - mv_0 = -2mv_0$
  - Particle adds $2mv_0$ momentum to wall
- How many hits in Δt time?
  - Must travel $2L$ before returning to wall
  - Covers this distance in $T = \frac{2L}{|v_0|}$
  - Hits $\frac{\Delta t}{T} = \frac{\Delta t|v_0|}{2L}$ times
  - Total momentum exchange: $\Delta p = \left(\frac{\Delta t|v_0|}{2L}\right)(2mv_0) = \frac{\Delta t m v_0^2}{L}$ times
  - Force: $F = \frac{\Delta p}{\Delta t} = \frac{mv_0^2}{L}$

2.2 Macroscopic

- N particles
- Total force
  - Force per particle: $F = \frac{mv^2}{L}$
  - Total force: $F = \sum_i \frac{mv_i^2}{L} = \frac{m}{L}\sum_i v_i^2 = \frac{mN\overline{v}^2}{L}$
  - *Average* velocity (root mean square, L_2 norm): $\overline{v} = \sqrt{\frac{1}{N}\sum_i v_i^2}$
- Kinetic energy
  - $KE = \sum_i \frac{1}{2}mv_i^2 = \frac{mN\overline{v}^2}{2} = \frac{L}{2}F$
  - Energetic particles apply more force
- Note that we care about the distribution $\overline{v}$ of velocities

3 Three dimensions

- Velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$
- Box dimensions: L_x, L_y, L_z.
- Face areas: $A_x = L_y L_z$, $A_y = L_x L_z$, $A_z = L_x L_y$
- Volume: $V = L_x L_y L_z$
- Collision
  - Velocity before: $\mathbf{v}_0 = \begin{pmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{pmatrix}$
  - Velocity after: $\mathbf{v}_1 = \begin{pmatrix} v_{x1} \\ v_{y1} \\ v_{z1} \end{pmatrix}$
  - No energy loss: $\frac{1}{2}m\mathbf{v}_0 \cdot \mathbf{v}_0 = \frac{1}{2}m\mathbf{v}_0 \cdot \mathbf{v}_0$ or $v_{x0}^2 + v_{y0}^2 + v_{z0}^2 = v_{x1}^2 + v_{y1}^2 + v_{z1}^2$
  - Assume hits a wall in x direction
  - Assume collision applies normal force only
    * Velocity after: $\mathbf{v}_1 = \begin{pmatrix} v_{x1} \\ v_{y0} \\ v_{z0} \end{pmatrix}$
    * $v_{x0}^2 = v_{x1}^2$
    * $v_{x1} = -v_{x0}$
  - Each dimension is independent of the others
- Pick a direction (x)
  - Force: $F_x = \frac{mN\overline{v_x}^2}{L_x}$
  - $\overline{v_x} = \sqrt{\frac{1}{N}\sum_i v_{ix}^2}$
  - Pressure: $P_x = \frac{F_x}{A_x} = \frac{mN\overline{v_x}^2}{L_x L_y L_z} = \frac{mN\overline{v_x}^2}{V}$
- Isotropic
  - Pressure is the same on all sides of container
  - $P_x = P_y = P_z$
  - $\overline{v_x} = \overline{v_y} = \overline{v_z}$
  - $PV = mN\overline{v_x}^2$
- Energy
  - $KE = \sum_i \frac{1}{2}m\mathbf{v}_i \cdot \mathbf{v}_i = \frac{m}{2}\sum_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{m}{2}(\sum_i v_{ix}^2 + v_{iy}^2 + v_{iz}^2) = \frac{mN}{2}(\overline{v_x}^2 + \overline{v_y}^2 + \overline{v_z}^2)$
  - Isotropic: $KE = \frac{3}{2}mN\overline{v_x}^2 = \frac{3}{2}PV$