Math 142-2, Final

Your name: ________________________

Student ID: ________________________

Instructions

This exam is closed book. No notes, books, electronic devices, or other resources are permitted on this exam. Be sure to write your name and student ID number at the top of this page. Scratch paper will provided for you to work out problems and write your answers. When you finish the exam, please staple all of the scratch paper that you have written on (even if it does not contain answers) to this question sheet when you turn in your exam.

Problem 1

A fictitious organism happens to come in three types (similar to how many animals come in male and female), which we can label A, B, and C. These organisms have very peculiar reproduction habits.

- Type A and type B can together reproduce to form type C.
- Type B and type C can together reproduce to form type A.
- Type C and type A can together reproduce to form type B.
- Two members of the same type reproduce to form offspring of the same type.
- These organisms are unaware that they come in three types, as all members look and behave alike.
- The organisms adjust their birth rates so as to maintain a constant total population T.

All types have the same death rates. Devise a model that describes the populations $A(t)$, $B(t)$, and $C(t)$ and is consistent with the above observations. *(5 points)*

Problem 2

Assume that $u = u_{\max}(1 - \rho/\rho_{\max})$.

(a) Show that if $\rho(x, 0) = \rho_{\max} - \rho(a - x, 0)$ then $\rho(x, t) = \rho_{\max} - \rho(a - x, t)$ for all $t \geq 0$. *(5 points)*

(b) Let $\rho(x, 0) = \frac{\rho_{\max}}{2}(1 + \sin(x))$ for the remaining parts of this problem. At $t = 0$, where will the cars move slowest? *(5 points)*

(c) At $t = 0$, where do the traffic waves move slowest? *(5 points)*

(d) When will the first shocks form? *(5 points)*

(e) Where will the first shocks form? *(5 points)*

(f) With what velocity will those shocks move? *(5 points)*

(g) For how long S will there be stopped cars? *(5 points)*

(h) If T is the time at which the first shock forms, sketch the traffic density for $-2\pi \leq x \leq 2\pi$ at $t = 0$, $t = \frac{T}{2}$, $t = T$, $t = S$, $t = 4T$, and $t \to \infty$. Do not attempt to solve the PDE analytically. *(5 points)*

Problem 3

The PDE $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$ is called the inviscid Burgers' equation. Because of its very simple and relatively intuitive behavior, it is classically studied as a way of understanding how PDEs behave, studying shocks and rarefactions, and testing numerical methods.

(a) Write this PDE in conservation form $U_t + F_x = 0$. What is the flux? What is the conserved quantity? *(5 points)*

(b) What is the local characteristic velocity? *(5 points)*

(c) If $v = \rho$, this PDE can be thought of as the equation for conservation of cars with some traffic following model $u(\rho)$. What is this traffic following model $u(\rho)$? Is this model reasonable? *(5 points)*

(d) Show that the more general transformation $v = a\rho + b$ allows this PDE to be transformed into the equation for conservation of cars with the traffic following model $u = u_{\max}(1 - \rho/\rho_{\max})$; find a and b. In other words, conservation of cars with this simple car following model behaves exactly the same as Burgers' equation; understanding how one evolves immediately provides insight into the behavior of the other. *(5 points)*

Problem 4

For each system below, determine the equilibrium points and classify each as stable, neutrally stable, or unstable. *(5 points each)*

(a) $\dfrac{dx}{dt} = -2x$

(b) $\dfrac{dx}{dt} = -2y \qquad \dfrac{dy}{dt} = 2x$

(c) $\dfrac{dx}{dt} = -2z \qquad \dfrac{dy}{dt} = 2x \qquad \dfrac{dz}{dt} = 2y$

(d) $\dfrac{dx}{dt} = e^{xy} \qquad \dfrac{dy}{dt} = x^2 + y^2 - 1$

(e) $\dfrac{dx}{dt} = e^{xy} - 1 \qquad \dfrac{dy}{dt} = x^2 - y^2 - 1$

Problem 5

Consider the model (where $x(t)$ and $y(t)$ are any real numbers)

$$\frac{dx}{dt} = xy \qquad \frac{dy}{dt} = y^2 - 1$$

(a) Locate the equilibria and classify each as {stable, neutrally stable, unstable} and {node, saddle point, spiral/loop}. *(5 points)*

(b) Plot the phase plane for this model. Be sure to include the equilibria, isoclines with arrows, and sample trajectories. *(5 points)*