1 Local existence and uniqueness theorem

Let $f(x, y)$ be

- Continuous as a function of two variables
- Lipschitz continuous in y: $|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$

in the rectangular region R defined by $x \in [a, b]$, $y \in [c, d]$. If $x_0 \in (a, b)$, $y_0 \in (c, d)$, then the ODE

$$y' = f(x, y) \qquad y(x_0) = y_0 \tag{1}$$

has exactly one solution $y = y(x)$ on the interval $x \in [x_0 - h, x_0 + h]$ for some $h > 0$.

2 Proof outline

1. Choose h.
2. The original ODE (1) is equivalent to
$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{2}$$
3. Define the sequence $y_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$, where $y_0(x) = y_0$.
4. The sequence $y_n(x)$ converges to a function $y(x)$.
5. The function $y(x)$ is continuous.
6. The function $y(x)$ satisfies (2).
7. If another function $\tilde{y}(x)$ satisfies (2), then $\tilde{y}(x) = y(x)$.

3 Part 1

The major complication with the proof of the local theorem compared with the global one is that the guarantees on $f(x, y)$ only apply inside the rectangle R. This forces us to limit our focus to an interval $x \in [x_0 - h, x_0 + h]$, where we can choose h to have the properties we need to make the proof work out.

First, note that $|f(x, y)|$ is continuous on the closed and bounded region R and thus has a maximum. We can now say $|f(x, y)| \leq M$ for some M. We choose h such that:

$$[x_0 - h, x_0 + h] \subseteq [a, b] \tag{3}$$
$$[y_0 - Mh, y_0 + Mh] \subseteq [c, d] \tag{4}$$
$$Kh < 1 \tag{5}$$

Note that the first two can be achieved for sufficiently small h, since (x_0, y_0) is in the *interior* of R, not on its boundary. These requirements were derived by looking at what assumptions are needed for the arguments in the proof to work. Let $S = [x_0 - h, x_0 + h] \times [y_0 - Mh, y_0 + Mh]$ be the restricted rectangular region, noting that $S \subseteq R$.

4 Part 2

The argument made in the global case is equally valid in the local case, except that I must now be careful to ensure $y(x) \in [c, d]$.

First, I would like to show that, restricted to $x \in [x_0 - h, x_0 + h]$, a solution to (1) lies in S. I will show this by contradiction, starting with the assumption that $|y(x_1) - y_0| > Mh$ for some $x_1 \in [x_0 - h, x_0 + h]$. Using continuity of $y(x)$, we are also able to show that we may choose $x_2 \in (x_0 - h, x_0 + h)$ so that

- $|y(x_2) - y_0| = Mh$
- $|y(x) - y_0| < Mh$ for $|x - x_0| < |x_2 - x_0| < h$

The mean value theorem guarantees the existence of a ξ between x and x_0 such that

$$\left|\frac{y(x_2) - y(x_0)}{x_2 - x_0}\right| = |y'(\xi)| = |f(\xi, y(\xi))| \leq M.$$

Note $|\xi - x_0| < |x_2 - x_0|$, so that $|y(\xi) - y_0| < Mh$. Since $(\xi, y(\xi)) \in S$, the assumptions on $f(x, y)$ apply. Finally

$$\left|\frac{y(x_2) - y(x_0)}{x_2 - x_0}\right| > \frac{|y(x_2) - y(x_0)|}{h} = \frac{Mh}{h} = M$$

leads to the necessary contradiction. It follows that $x \in [x_0 - h, x_0 + h]$ implies $|y(x) - y_0| \leq Mh$, so that the solution lies in S.

Since the solution is limited to S, the arguments used to prove the case (1) implies (2) in the global case remain valid here.

The following logic can now be used to show that the other case (2) implies (1) is covered.

1. We construct a solution y to the integral equation, which we will show later is in S.
2. This y satisfies the assumptions used to show that it satisfies the ODE.
3. Assume there is another solution $\tilde{y}$ to the ODE. We show above that this lies in S.
4. This must also satisfy the integral equation.
5. Since both y and $\tilde{y}$ satisfy the integral equation and are in S, we can show using the argument below that $\tilde{y} = y$.

Note that this sequence of logic suffices to prove the local existence-uniqueness theorem.

Note that I have not ruled out the existence of a solution to the integral equation, distinct from constructed solution y and not in S, which is also not a solution to the ODE. Although this seems unsatisfying, I don't need to worry about this, since it does not affect the validity of the theorem I am trying to prove.

5 Part 3: Picard iteration

Define the sequence

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt, \qquad y_0(x) = y_0. \tag{6}$$

This is called Picard iteration. For this sequence to be useful, we must show that it lies in S. If $x \in [x_0 - h, x_0 + h]$ then

$$|y_n(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_{n-1}(t))\, dt\right| \leq Mh \tag{7}$$

Note that this must be used inductively, since it relies on $y_{n-1}(t) \in [c, d]$. The statement is trivial for $n = 0$, which allows the bound above to be applied inductively. All of the Picard iterates are in rectangle S.

6 Part 4

I need to show that $y_n(x) \to y(x)$ as $n \to \infty$. The proof from the global case can be used here. The approach that the book takes follows the more historical approach. Although the book does not mention this, this proof amounts to (a) showing that Picard iteration is a *contraction*, and then (b) proving the Banach fixed-point theorem. The proof is somewhat simpler than this, however, since more context is available.

6.1 Bounding consecutive approximations

Note that $|y_0(x) - y_1(x)|$ is continuous on the closed and bounded interval $[a, b]$, so it must take on a maximum value in this interval. Thus, for some M, I can write $|y_0(x) - y_1(x)| \le M$. As before, the key to bounding subsequent intervals is to use Lipschitz continuity to relate consecutive differences using (assuming $x > x_0$)

$$\begin{aligned}
|y_n(x) - y_{n-1}(x)| &= \left| \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt - \int_{x_0}^{x} f(t, y_{n-2}(t))\, dt \right| && (8)\\
&= \left| \int_{x_0}^{x} f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))\, dt \right| && (9)\\
&\le \int_{x_0}^{x} |f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))|\; dt && (10)\\
&\le \int_{x_0}^{x} K|y_{n-1}(t) - y_{n-2}(t)|\, dt && (11)\\
&= K \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)|\, dt && (12)
\end{aligned}$$

This leads us to the following sequence of differences

$$\begin{aligned}
|y_1(x) - y_0(x)| &\le M \\
|y_2(x) - y_1(x)| &\le K \int_{x_0}^{x} |y_1(t) - y_0(t)|\, dt \le K \int_{x_0}^{x} M\, dt = MKh \\
|y_3(x) - y_2(x)| &\le K \int_{x_0}^{x} |y_2(t) - y_1(t)|\, dt \le K \int_{x_0}^{x} MKh\, dt = M(Kh)^2 \\
|y_4(x) - y_3(x)| &\le K \int_{x_0}^{x} |y_3(t) - y_2(t)|\, dt \le K \int_{x_0}^{x} M(Kh)^2\, dt = M(Kh)^3 \\
&\vdots \\
|y_n(x) - y_{n-1}(x)| &\le K \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)|\, dt \le K \int_{x_0}^{x} M(Kh)^{n-2}\, dt = M(Kh)^{n-1}
\end{aligned}$$

This bound was obtained assuming $x > x_0$, since otherwise many of the integrals are negative. I could repeat the argument with $x < x_0$ and get the same result.

6.2 Convergence

I can express the terms in the sequence as a telescoping sum

$$y_n(x) = y_0 + \sum_{k=1}^{n} (y_k(x) - y_{k-1}(x)). \tag{13}$$

If $y_n(x) \to y(x)$, then

$$y(x) = y_0 + \sum_{k=1}^{\infty} (y_k(x) - y_{k-1}(x)). \tag{14}$$

Note that this is just the definition of an infinite series. The limit exists if and only if the series converges. Since

$$\sum_{k=1}^{\infty} |y_k(x) - y_{k-1}(x)| \leq \sum_{k=1}^{\infty} M(Kh)^{k-1} = \frac{M}{1-Kh} \tag{15}$$

converges provided $Kh < 1$, $y_n(x) \to y(x)$. Now we see the reason for the additional requirement $Kh < 1$ that we imposed when choosing h.

One more piece is needed before we continue. Since $|y_n(x) - y_0| \leq Mh$ for all n, we must have $|y(x) - y_0| \leq Mh$, so that y lies in S.

7 Part 5

This is left as a homework assignment.

8 Part 6

This proof is the same as in the global case.

9 Part 7

Next, lets consider uniqueness. Assume y and $\tilde{y}$ are two solutions to (2). Let $Q = \max_x |\tilde{y}(x) - y(x)|$.

$$\begin{aligned}
Q &= \max_x |\tilde{y}(x) - y(x)| && (16)\\
&= \max_x \left| \int_{x_0}^{x} f(t, \tilde{y}(t))\, dt - \int_{x_0}^{x} f(t, y(t))\, dt \right| && (17)\\
&\leq \max_x \int_{x_0}^{x} |f(t, \tilde{y}(t)) - f(t, y(t))|\, dt && (18)\\
&\leq K \max_x \int_{x_0}^{x} |\tilde{y}(t) - y(t)|\, dt && (19)\\
&\leq K \max_x \int_{x_0}^{x} Q\, dt && (20)\\
&\leq K \max_x Qh && (21)\\
&\leq KhQ && (22)
\end{aligned}$$

Noting $Kh < 1$ and $Q \geq 0$, we conclude that $Q = 0$. Thus, $\tilde{y} = y$. Since all solutions are equal, the solution is unique.